Composition operators induced by universal covering maps

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Preliminaries

- $\mathbb{D} = \{z \colon |z| < 1\}$
- H^p $(1 \leq p < \infty)$ the Hardy space, $f \in H^p$ if and only if

$$\|f\|_p^p = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty$$

- If $\phi \colon \mathbb{D} \to \mathbb{D}$ is holomorphic then the composition operator C_ϕ is

$$\begin{aligned} C_\phi \quad : \quad & H^p \to H^p \\ & f \mapsto f \circ \phi \end{aligned}$$

Boundedness of C_ϕ

Theorem (Littlewood's subordination theorem (1925))

Suppose that $\phi\colon \mathbb{D} \to \mathbb{D}$ is univalent and holomorphic, $\phi(0) = 0$. If $f \in H^p$ $(1 \leq p < \infty)$ then

$$\int_0^{2\pi} |f(\phi(re^{i\theta}))|^p d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

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So

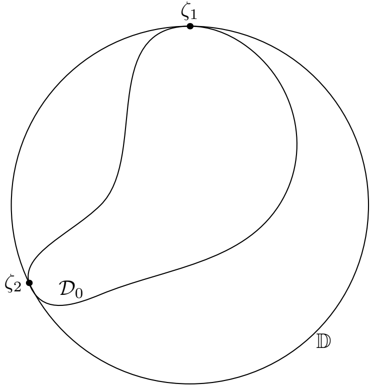
$$C_\phi\colon H^p \to H^p$$

is a bounded operator.

Compactness of C_ϕ

If $\phi\colon \mathbb{D} \to \mathcal{D}_0$ is univalent then C_ϕ is compact if and only if

$$\lim_{|z|\to 1} \frac{1-|\phi(z)|}{1-|z|} = \infty.$$



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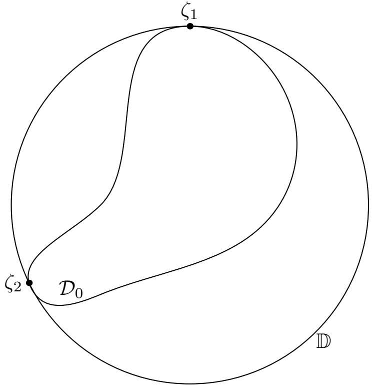
The angular derivative does not exist anywhere.

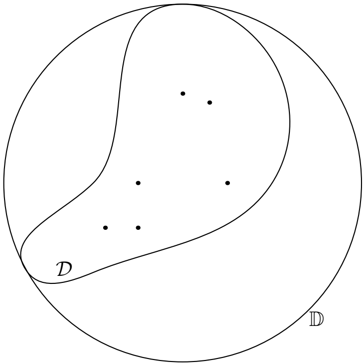


Compactness of C_ϕ and $\phi(\mathbb{D})$

Consider instead domains of the form

$$\mathcal{D} = \mathcal{D}_0 \backslash \{p_1, p_2, \ldots, p_n\}$$

where p_k, $k = 1, 2, \ldots n$ are isolated points in $\mathcal{D}_0$.

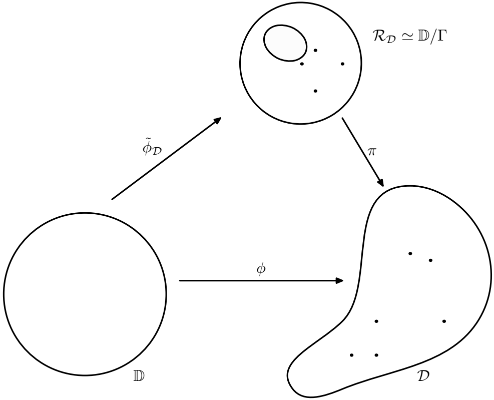


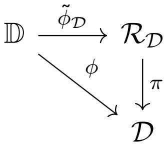
The uniformization theorem

- For simply connected $\mathcal{D}_0$ we have the Riemann mapping theorem
- There is a 'unique' univalent mapping ψ of $\mathbb{D}$ onto $\mathcal{D}_0$.

- For domains $\mathcal{D} = \mathcal{D}_0 \backslash \{p_1, p_2, \ldots, p_n\}$ we must employ the uniformization theorem
- There is a holomorphic universal covering map of $\mathbb{D}$ onto $\mathcal{D}$.

The universal covering map





Example

- Riemann mapping from $\mathbb{D}$ to $\mathbb{D}$ is $\psi(z) = z$
- A universal covering map from $\mathbb{D}$ to $\mathbb{D}\backslash\{0\}$ is

$$\phi(z) = \exp\left(-\frac{1+z}{1-z}\right)$$

- Note ϕ is an inner function with

$$\lim_{r\to 1^-} |\phi(re^{i\theta})| = \begin{cases} 0, & \theta = 0 \\ 1, & \text{otherwise} \end{cases}$$

Multivalent functions

Definition (Nevanlinna's counting function)

For any function $\phi\colon \mathbb{D} \to \mathbb{D}$

$$\mathcal{N}_\phi(w) = \begin{cases} \displaystyle\sum_{z\colon\, \phi(z)=w} \log \frac{1}{|z|} & w \in \phi(\mathbb{D}) \\ 0 & w \in \mathbb{D}\backslash\overline{\phi(\mathbb{D})} \end{cases}$$

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Theorem (Shapiro's compactness criterion)

C_ϕ is compact on H^p if and only if

$$\lim_{|w|\to 1} \frac{\mathcal{N}_\phi(w)}{\log 1/|w|} = 0$$

For our universal covering map

$$\phi\colon \mathbb{D} \to \mathcal{D} = \mathcal{D}_0 \setminus \{p_1, \ldots, p_n\}$$

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\tilde{\phi}_{\mathcal{D}}} & \mathcal{R}_{\mathcal{D}} \\ & \searrow^{\phi} & \downarrow \pi \\ & & \mathcal{D} \end{array}$$

we have:

- $\mathcal{R}_{\mathcal{D}} \simeq \mathbb{D}/\Gamma$ for Γ a torsion-free Fuchsian group
- $\Lambda(\Gamma)$ – the limit set of Γ – satisfies $\Lambda(\Gamma) \subsetneq \partial\mathbb{D}$
- $\phi^{-1}(w)$ is a Γ-orbit
- $z_1, z_2 \in \phi^{-1}(w)$ if and only if $\exists h \in \Gamma$ such that $z_1 = h(z_2)$.

Main results

Theorem

Suppose that $\mathcal{D} = \mathcal{D}_0 \backslash \{p_1, \ldots, p_n\}$ *and* ϕ *is a universal covering of* $\mathbb{D}$ *onto* $\mathcal{D}$*. Then* C_ϕ *is compact on* H^p, $1 \leq p < \infty$, *if and only if*

$$\lim_{z \to \zeta} \frac{1 - |\phi(z)|}{1 - |z|} = \infty$$

for all $\zeta \in \partial\mathbb{D}$.

Main results

Theorem

Suppose that $\mathcal{D} = \mathcal{D}_0 \backslash \{p_1, \ldots, p_n\}$ and ϕ is a universal covering of $\mathbb{D}$ onto $\mathcal{D}$. Then C_ϕ is compact on H^p, $1 \leq p < \infty$, if and only if

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Theorem

Suppose that $\mathcal{D} = \mathcal{D}_0 \backslash \{p_1, \ldots, p_n\}$, ϕ is a universal covering of $\mathbb{D}$ onto $\mathcal{D}$, and ψ is the univalent Riemann mapping of $\mathbb{D}$ onto $\mathcal{D}_0$. Then C_ϕ is compact on H^p, $1 \leq p < \infty$, if and only if C_ψ is.

Proof of results

Definition (The Poincare series)

For a Fuchsian group Γ and $z, w \in \mathbb{D}$

$$\rho_\Gamma(z, w; s) = \sum_{g \in \Gamma} \exp -s d_{\mathbb{D}}(z, g(w))$$

where

$$d_{\mathbb{D}}(z, w)$$

is the hyperbolic distance in $\mathbb{D}$.

Lemma

There are constants c_1 and c_2 such that for z with $w = \phi(z)$ suitably chosen

$$c_1 \rho_\Gamma(0, z; 1) \leq \mathcal{N}_\phi(w) \leq c_2 \rho_\Gamma(0, z; 1)$$

Proof of result

Lemma

C_ϕ is compact on H^p if and only if for all $\zeta \notin \Lambda(\Gamma)$

$$\lim_{z \to \zeta} \frac{\rho_\Gamma(0, z; 1)}{1 - |\phi(z)|} = 0$$

Lemma

If Γ is finitely generated then there are constants c_1 and c_2 such that for z close enough to $\partial\mathbb{D}\backslash\Lambda(\Gamma)$

$$c_1(1 - |z|^2) \le \rho_\Gamma(0, z; 1) \le c_2(1 - |z|^2)$$

Proof of main result

1. C_ϕ compact if and only if for all $\zeta \notin \Lambda(\Gamma)$

$$\lim_{z \to \zeta} \frac{\rho_\Gamma(0, z; 1)}{1 - |\phi(z)|} = \lim_{z \to \zeta} \frac{1 - |z|}{1 - |\phi(z)|} = 0$$

2. $\Lambda(\Gamma)$ consists of:
   - fixed points of parabolic elements (correspond to p_i)
   - points of approximation (where orbits converge non-tangentially and therefore ϕ is fixed on a sequence converging non-tangentially)

Extension of work

- These results can be extended to general (finitely) multiply connected domains
- Widely applicable in the study of composition operators since the Nevanlinna counting function appears often
- The characterisation of Schatten class composition operators in particular employs $\mathcal{N}_\phi$.