Compact composition operators with symbol an universal covering map

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Abstract

In this paper we study composition operators, C_ϕ, acting on the Hardy spaces that have symbol, ϕ, a universal covering map of the disk onto a finitely connected domain of the form $\mathcal{D}_0 \backslash \{p_1, \cdots, p_n\}$, where $\mathcal{D}_0$ is simply connected and p_i, $i = 1, \ldots, n$, are distinct points in the interior of $\mathcal{D}_0$. We consider, in particular, conditions that determine compactness of such operators and demonstrate a link with the Poincare series of the uniformizing Fuchsian group. We show that C_ϕ is compact if, and only if ϕ does not have a finite angular derivative at any point of the unit circle, thereby extending the result for univalent and finitely multivalent ϕ.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ be the unit disk in the complex plane, then the Hardy space H^p, $1 \leq p < \infty$, is defined to be the Banach space of functions holomorphic in $\mathbb{D}$ with norm

$$\|f\|_p^p = \lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

The limit here is guaranteed by the fact that the integral mean is increasing in r. The standard text for the theory of Hardy spaces is [6].

Given a holomorphic map $\phi \colon \mathbb{D} \to \mathbb{D}$ we define the composition operator

$$C_\phi \colon f \to f \circ \phi.$$

The study of composition operators acting on function spaces has received much attention over the last four decades. The central theme of this work is to understand how operator theoretic properties of composition operators are related to geometric or analytic properties of their inducing functions. Of central importance in this area is a result of Shapiro, [10], which describes the essential norm of a composition operator in terms of the Nevanlinna counting function of its inducing holomorphic map. The Nevanlinna counting function is known explicitly in a number of situations, for example for inner functions, univalent functions and finitely multivalent functions.

In this paper we study composition operators with symbol a universal covering map of the unit disk onto a finitely connected domain, in this case the Nevanlinna counting function can be estimated precisely by properties of the underlying Fuchsian group. We will provide all the preliminary definitions in section 2.

We consider throughout this article domains of the form

$$\mathcal{D} = \mathcal{D}_0 \backslash \{p_1, \ldots, p_n\} \qquad n \geq 1 \tag{1}$$

where $\mathcal{D}_0$ is a simply connected domain contained in $\mathbb{D}$ and $p_1, \ldots, p_n$ are distinct, isolated points in the interior of $\mathcal{D}_0$. We will study composition operators whose symbol ϕ is the universal covering map of $\mathbb{D}$ onto $\mathcal{D}$.

For a Fuchsian group Γ we define the limit set $\Lambda(\Gamma)$ to be the set of accumulation points of orbits of points in $\mathbb{D}$ by functions in Γ. The Poincare series for Γ of order s is

$$\rho_\Gamma(z, w; s) = \sum_{g \in \Gamma} \exp -s d_{\mathbb{D}}(z, g(w)) \tag{2}$$

where $d_{\mathbb{D}}(z, w)$ is the hyperbolic distance from z to w in $\mathbb{D}$.

It is known that there is a critical exponent, $\delta(\Gamma)$ such that the Poincare series converges for all $s < \delta(\Gamma)$ but diverges for all $s > \delta(\Gamma)$. For finitely generated Fuchsian groups

$$\delta(\Gamma) = \dim(\Lambda(\Gamma)),$$

the Hausdorff dimension of the limit set of Γ.

A simple calculation shows that if Γ is elementary and generated by a parabolic element then

$$\delta(\Gamma) = 1/2.$$

If Γ is non-elementary and contains a parabolic element then Beardon showed in [3] that

$$\delta(\Gamma) > 1/2$$

and if Γ is finitely generated and of the second kind then

$$\delta(\Gamma) < 1. \tag{3}$$

Our first result links the compactness of a composition operator to the growth of the universal covering map with respect to the Poincare series.

Theorem 1. *Let $\mathcal{D}$ be a domain in $\mathbb{D}$ defined by (1) and suppose that ϕ is a universal covering map of $\mathbb{D}$ onto $\mathcal{D}$.*

Let Γ be the Fuchsian group that uniformizes $\mathcal{D}$, then C_ϕ is compact on H^p, $1 \leq p < \infty$, if and only if for each $\zeta \in \partial\mathbb{D}\backslash\Lambda(\Gamma)$

$$\lim_{z\to\zeta} \frac{\rho_\Gamma(0, z; 1)}{1 - |\phi(z)|} = 0 \tag{4}$$

Note that the hypothesis implies that Γ is finitely generated and so (4) is well defined by (3).

An important geometric quantity that has proved useful in describing compactness of composition operators has been the angular derivative, see [5] or [11]. A holomorphic mapping $\phi\colon \mathbb{D} \to \mathbb{D}$ has a finite angular derivative $|\phi'(\zeta)|$ for $\zeta \in \partial\mathbb{D}$ if

$$\liminf_{z\to\zeta} \frac{1 - |\phi(z)|}{1 - |z|} < \infty.$$

The existence of a finite angular derivative implies a number of well behaved mapping properties of ϕ near ζ, a good reference for this is [2], see also the overviews in [5] and [11]. Note that if an angular derivative exists then, in particular, $\lim_{z\to\zeta} |\phi(z)| = 1$, where the limit is non-tangential.

To appreciate the importance of this quantity it is known that, for ϕ univalent, C_ϕ is compact if, and only if

$$\lim_{|z|\to 1} \frac{1 - |\phi(z)|}{1 - |z|} = \infty, \tag{5}$$

or, equivalently, ϕ does not have a finite angular derivative at any point on $\partial\mathbb{D}$. For arbitrary ϕ it was shown in [12] that if C_ϕ is compact then ϕ does not have a finite angular derivative at any point on $\partial\mathbb{D}$, however in general it is not difficult to find counterexamples to the converse. For example no inner function induces a compact operator but there are inner functions with no angular derivative at any point on $\partial\mathbb{D}$, see [11, §10.2].

We generalise the above result to the current setting.

Theorem 2. *Suppose that $\mathcal{D}$ is defined by (1) and ϕ is a universal covering of $\mathbb{D}$ onto $\mathcal{D}$. Then C_ϕ is compact on H^p, $1 \leq p < \infty$, if and only if*

$$\lim_{z\to\zeta} \frac{1 - |\phi(z)|}{1 - |z|} = \infty$$

for all $\zeta \in \partial\mathbb{D}$.

It follows that the counterexamples to Shapiro and Taylor's result cannot come from universal covering maps of finitely connected domains.

This result begins to demonstrate the link between the compactness of C_ϕ and the geometry of the image domain. In fact, as a consequence of the previous theorem and properties of inner functions that we will discuss later, we can develop this idea further.

Theorem 3. *Suppose that $\mathcal{D}$ is defined by (1), ϕ is a universal covering of $\mathbb{D}$ onto $\mathcal{D}$, and ψ is the univalent Riemann mapping of $\mathbb{D}$ onto $\mathcal{D}_0$. Then C_ϕ is compact on H^p, $1 \leq p < \infty$, if and only if C_ψ is.*

There are a number of geometric interpretations of the existence of an angular derivative for univalent functions that can now be applied to $\mathcal{D}_0$ that will ensure compactness of C_ϕ. We will not list these here but many of these cane be found in [7] and throughout the literature on compact composition operators.

2 Preliminaries

In this section we will state and discuss Shapiro's characterisation of compact composition operators, followed by an short introduction to the relevant theory of universal covering maps and Fuchsian groups.

Recall the Calkin algebra for H^p is the algebra $B(H^p)/B_0(H^p)$ where $B(H^p)$ is the algebra of bounded linear operators mapping H^p to H^p, and $B_0(H^p)$ is the corresponding ideal of compact operators in $B(H^p)$. The essential norm of an operator T, written $\|T\|_e$ is the norm of T in the Calkin algebra. The essential norm measures the distance, in the norm induced metric, to the compact operators,

$$\|T\|_e = \inf_{K \in B_0(H^p)} \|T - K\|.$$

Shapiro's result, [10], provides a formula for the essential norm of C_ϕ that describes precisely its relationship with the inducing function ϕ. In order to state Shapiro's result we define the Nevanlinna counting function for ϕ to be

$$\mathcal{N}_\phi(w) = \begin{cases} \displaystyle\sum_{z:\ \phi(z)=w} \log \frac{1}{|z|} & w \in \phi(\mathbb{D}) \\ 0 & w \in \mathbb{D} \backslash \overline{\phi(\mathbb{D})} \end{cases}$$

It is known and relatively easy to estimate $\mathcal{N}_\phi$ when ϕ is finitely valent. Shapiro proved that

$$\|C_\phi\|_e^2 = \limsup_{|w| \to 1} \frac{\mathcal{N}_\phi(w)}{\log \dfrac{1}{|w|}}. \tag{6}$$

In particular, C_ϕ is compact on H^p if and only if

$$\lim_{|w|\to 1} \frac{\mathcal{N}_\phi(w)}{\log \dfrac{1}{|w|}} = 0.$$

An inner function is a bounded holomorphic function, I, on $\mathbb{D}$ for which

$$\lim_{r\to 1} |I(re^{i\theta})| = 1$$

for almost every $\theta \in [0, 2\pi)$ with respect to Lebesugue measure. It is known that, outside a set of 2-dimensional Lebesgue measure 0, an inner function I satisfies

$$\mathcal{N}_I(w) = \log \left| \frac{I(0) - w}{1 - \overline{I(0)}w} \right|.$$

An example of an inner function that is relevant to the current work is the function

$$z \mapsto \exp\left(-\frac{1+z}{1-z}\right)$$

that maps $\mathbb{D}$ conformally onto $\mathbb{D}\backslash\{0\}$. It is notable that the radial limit of this function along the positive real axis is 0, whereas all other radial limits have modulus 1. It has infinite angular derivative at 1 but has finite angular derivative elsewhere on $\partial\mathbb{D}$. This is the universal covering map of $\mathbb{D}$ onto $\mathbb{D}\backslash\{0\}$.

In this paper we examine how Shapiro's characterisation of compact composition operators may be interpreted when ϕ is a universal covering map. We will cover the prerequisite details required here in order to fix notation and relevant ideas. First recall that the hyperbolic metric, [2], on $\mathbb{D}$ is defined by

$$d_{\mathbb{D}}(z, w) = \inf \int_\gamma \frac{2}{1 - |z|^2} |dz|$$

where the infimum is taken over all smooth curves γ connecting z to w in $\mathbb{D}$. The constant 2 is required to ensure that the Gaussian curvature of the metric is equal to -1 throughout $\mathbb{D}$, it is often omitted in the literature. This metric is so called because it induces Poincare's disk model of hyperbolic space where geodesics are arcs of circles orthogonal to the unit circle or radii. In particular, we have that

$$d_{\mathbb{D}}(0, w) = \log \frac{1 + |w|}{1 - |w|}. \tag{7}$$

Automorphisms of $\mathbb{D}$ are of the form

$$z \mapsto \lambda \frac{a - z}{1 - \overline{a}z}$$

where $|\lambda| = 1$ and $a \in \mathbb{D}$, and are isomorphisms in the hyperbolic metric. They are classified as elliptic, parabolic or hyperbolic according to whether they have a fixed point in $\mathbb{D}$, a fixed point in $\partial\mathbb{D}$, or 2 fixed points in $\partial\mathbb{D}$ respectively. The theory of automorphisms of $\mathbb{D}$ are covered in detail in [4] where many of the results concerning Fuchsian groups in this section may be found.

A group Γ of automorphisms of $\mathbb{D}$ may be considered a subspace of the topological space $GL_2(\mathbb{C})$, Γ is called a *Fuchsian Group* if it is discrete in the subspace topology. For any hyperbolic Riemann surface, $\mathcal{R}$, there is a Fuchsian group $\Gamma_{\mathcal{R}}$ that contains no elliptic elements such that $\mathcal{R}$ is homeomorphic to $\mathbb{D}/\Gamma$.

Given a domain $\mathcal{D} \subset \mathbb{D}$ there is a Riemann surface $\mathcal{R}_{\mathcal{D}}$ and a covering projection $\pi\colon \mathcal{R}_{\mathcal{D}} \to \mathcal{D}$. Since $\mathcal{R}_{\mathcal{D}}$ is conformally equivalent to $\mathbb{D}$ by the uniformization theorem we may find a $\tilde{\phi}_{\mathcal{D}}\colon \mathbb{D} \to \mathcal{R}_{\mathcal{D}}$ so that the mapping

$$\phi = \pi \circ \tilde{\phi}_{\mathcal{D}}$$

maps $\mathbb{D}$ conformally onto $\mathcal{D}$:

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\tilde{\phi}_{\mathcal{D}}} & \mathcal{R}_{\mathcal{D}} \\ & \searrow^{\phi} & \downarrow{\scriptstyle \pi} \\ & & \mathcal{D} \end{array}$$

ϕ is the *universal covering map* of $\mathcal{D}$ and is unique up to pre-composition with an automorphism of $\mathbb{D}$. It follows from the construction above that the inverse of $\phi(w)$ for any $w \in \mathcal{D}$ is the fiber over w and this is a Γ-orbit, i.e. is of the form $\Gamma(z) = \{g(z)\colon\, g \in \Gamma\}$.

A fundamental domain for the action of Γ on $\mathbb{D}$ is said to be locally finite if each compact subset of $\mathbb{D}$ meets only finitely many Γ-images of $\tilde{\mathcal{F}}$. $\mathcal{F}$ is locally finite if and only if the mapping

$$\theta\colon\, \tilde{\mathcal{F}} \cap \Gamma(z) \mapsto \Gamma(z)$$

is a homeomorphism from of $\tilde{\mathcal{F}}/\Gamma$ onto $\mathbb{D}/\Gamma$. Here $\tilde{\mathcal{F}}$ represents the relative closure of $\mathcal{F}$ in $\mathbb{D}$.

The Dirichlet fundamental polygon for Γ is defined for given $w \in \mathbb{D}$ as

$$D(w) = \bigcap_{g \in \Gamma, g \neq id} \{z \in \mathbb{D}\colon\, d_{\mathbb{D}}(z, w) < d_{\mathbb{D}}(z, g(w))\};$$

it is locally finite.

Finally for a Fuchsian group of the second kind the set of discontinuity is

$$\Omega(\Gamma) = \hat{\mathbb{C}} \backslash \Lambda(\Gamma),$$

where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This set is connected and the action of Γ can be extended canonically to $\Omega(\Gamma)$ where it acts discontinuously.

2.1 Examples

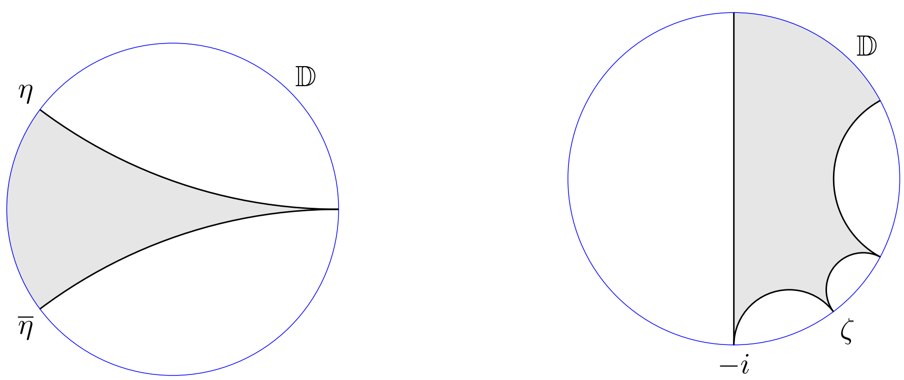

Figure 1: Fundamental domains for $n = 1$ and $n = 2$

$\underline{n = 1}$ In the case $n = 1$ the domain $\mathcal{D}$ is uniformized by an elementary Fuchsian group of the form

$$\Gamma = \langle \varrho \rangle,$$

with ϱ a parabolic disk automorphism. Suppose that ϱ has fixed point 1 then the Dirichlet domain $D(0)$ is shown on the left in Figure 1

The two sides of $\mathcal{F}$ in $\mathbb{D}$ are equivalent in $\mathbb{D}/\langle \varrho \rangle$. The free side of $\mathcal{F}$ is homeomorphic to $\partial \mathcal{D}_0$ (note the two endpoints of the free side are equivalent).

$\underline{n = 2}$ For the case $n = 2$ the domain $\mathcal{D}$ is conformally equivalent to the Riemann surface $\mathbb{D}/\Gamma$ where Γ is generated by two parabolic automorphisms, ϱ_1 and ϱ_2. A fundamental set for Γ is illustrated on the right in Figure 1, here we assume that the fixed points of ϱ_1 and ϱ_2 are ζ and $-i$. The point ζ can be determined from the geometry of $\mathcal{D}$, specifically the length of the closed hyperbolic geodesic separating the points p_1 and p_2 from the boundary of $\mathcal{D}_0$. This example is taken from [9] where a more detailed discussion is available.

3 Proof of Theorem 1

We will assume that $\partial\mathcal{D}_0 \cap \partial\mathbb{D} \neq \emptyset$, the result (and all main results) are trivially true if $\sup_\zeta |\phi(\zeta)| < 1$ in which case the angular derivative cannot exist anywhere.

Note first that the points $p_i, i = 1, \ldots, n$, are considered punctures in the Riemann surface and are therefore in one-to-one correspondence with the conjugacy class of parabolic elements in Γ, see [8, pages 214-216].

Let $\mathcal{F}$ be a locally finite fundamental domain for the action of Γ on $\mathbb{D}$. Then $\mathcal{F}$ can be chosen to be a finite sided convex polygon with one free side contained in $\partial\mathbb{D}$, for example we may take $\mathcal{F}$ to be a Dirichlet convex fundamental polygon.

Now $\tilde{\mathcal{F}}/\Gamma$ is homeomorphic to $\mathbb{D}/\Gamma$ so that we may define a branch of the inverse of ϕ on a subdomain of $\mathbb{D}$, ψ say, that maps this subdomain univalently onto $\mathcal{F}$.

Let I be the free side of $\mathcal{F}$ then as $|w| \to 1$ in $\mathcal{D}$, $z = \psi(w)$ tends to I. To see this we simply need to ensure that z does not converge to other boundary points of $\mathcal{F}$. To this end, suppose that Γ is non-elementary, then I is contained in an interval of discontinuity of Γ on $\partial\mathbb{D}$, γ say. If we let A be the hyperbolic geodesic in $\mathbb{D}$ with the same endpoints as γ then $A \cap \tilde{\mathcal{F}}$ is homeomorphic to the closed hyperbolic geodesic in $\mathcal{D}$ that separates $\partial\mathcal{D}_0$ from the points $p_1, \ldots, p_n$. Therefore A separates I from other corners of $\mathcal{F}$ and so as

$$w \to \partial\mathcal{D}\backslash\{p_1, \ldots, p_n\}, \qquad z \to I.$$

One can check the case $n = 1$ when Γ is elementary directly.

Assume then that $|z| > R > \frac{1}{2}$ for a given R. Then for $w \in \mathcal{D}$

$$\mathcal{N}_\phi(w) = \sum_{g\in\Gamma} \log \frac{1}{|g(z)|}.$$

Since Γ is discontinuous on $\mathbb{D}$ there are only finitely many $g \in \Gamma$ with $g(z) \in \{z\colon |z| \leq R\}$. Hence, using the inequality

$$\log\frac{1}{x} \leq 1 - x^2 \leq 2\log\frac{1}{x}, \qquad 1/2 < x < 1$$

we have that

$$\begin{aligned}
\mathcal{N}_\phi(w) &\leq C\sum_{g\in\Gamma}(1-|g(z)|)^2 \\
&\leq C\sum_{g\in\Gamma}\frac{1-|g(z)|}{1+|g(z)|} \\
&= C\sum_{g\in\Gamma}\exp -d_{\mathbb{D}}(0, g(z)) \\
&= C\rho_\Gamma(0, z; 1)
\end{aligned}$$

Similarly $\mathcal{N}_\phi(w) \geq C\rho_\Gamma(0, z; 1)$.

We have shown thus far that

$$\lim_{|w|\to 1} \frac{\mathcal{N}_\phi(w)}{\log \frac{1}{|w|}} = 0$$

if and only if

$$\lim_{z\to I} \frac{\rho_\Gamma(0, z; 1)}{1 - |\phi(z)|} = 0 \tag{8}$$

where the limit takes place in $\tilde{\mathcal{F}}$.

To show that this implies our result note that since Γ is discontinuous on $\Omega(\Gamma)$, we have that for any closed arc $J \subset \partial\mathbb{D}\backslash\Lambda(\Gamma)$ finitely many images of $\tilde{\mathcal{F}}$ under mappings in Γ cover J and we may apply (8) to each without difficulty using the automorphic property of ρ_Γ. Therefore the limit (4) is zero at any point in $\zeta \in J$ and, in particular at any point in $\partial\mathbb{D}\backslash\Lambda(\Gamma)$.

The converse, that (4) implies (8), is, of course, trivial.

4 Proof of Theorems 2 and 3

In order to prove this result we will require the following quantitative estimate of the Poincare series of index 1.

Lemma 1. *If* Γ *uniformizes a domain of the form (1) then for* $z \in D(0)$ *with* $|z|$ *close enough to* 1.

$$c_1 \exp -d_{\mathbb{D}}(0, z) \leq \rho_\Gamma(0, z, 1) \leq c_2 \exp -d_{\mathbb{D}}(0, z)$$

where c_1 *and* c_2 *are constants depending only on* Γ.

Remark: the left-hand inequality above holds for arbitrary $z \in \mathbb{D}$, and, in fact, this is the way we will use it.

Proof. We prove the left hand inequality first.

Let z, w be arbitrary and define

$$h(z, w) = \frac{|1 - \overline{z}w|^2}{(1 - |z|^2)(1 - |w|^2)}$$

then

$$\sum_{g\in\Gamma} h(z, g(w))^{-s} \leq \rho_\Gamma(z, w; s) \leq 2^s \sum_{g\in\Gamma} h(z, g(w))^{-s}.$$

Furthermore we have the estimate, for $z, w, v \in \mathbb{D}$,

$$h(z,w) \leq 4h(z,v)h(v,w).$$

To see this note that since $h(z,w) = h(\gamma(z), \gamma(w))$ for any automorphism γ of $\mathbb{D}$ we need only prove this for $v = 0$ and this is a straightforward calculation. Therefore we have

$$\begin{aligned}
\rho_\Gamma(0,z;1) &\geq 4^{-1} \sum_{g\in\Gamma} h(0,g(0))^{-1} h(g(0),g(z))^{-1} \\
&= 4^{-1} \sum_{g\in\Gamma} h(0,g(0))^{-1} h(0,z)^{-1} \\
&= \frac{1}{4}(1-|z|^2)\rho_\Gamma(0,0;1) \\
&\geq \frac{1}{4}\left(\frac{1-|z|}{1+|z|}\right)\rho_\Gamma(0,0;1)
\end{aligned}$$

It follows that

$$\rho_\Gamma(0,z;1) \geq c_1 \max_{g\in\Gamma} \exp -d_{\mathbb{D}}(0,g(z)).$$

This maximum is attained for $g(z) \in D(0)$ and so the left-hand inequality is proved.

To prove the right hand inequality, let $z \in D(0)$.

Let $0 < \delta < 1$, then we may define

$$\begin{aligned}
A &= \{g \in \Gamma \colon |z - g^{-1}(0)| < \delta \text{dist}(z, \Lambda(\Gamma))\} \\
B &= \Gamma \backslash A
\end{aligned}$$

Then we have

$$\rho_\Gamma(0,z;1) \leq \sum_{g\in A} \exp -d_{\mathbb{D}}(0,g(z)) + 2\sum_{g\in B} 1 - |g(z)|^2 \tag{9}$$

Since $z \in D(0)$ we have that $d_{\mathbb{D}}(0,g(z)) \geq d_{\mathbb{D}}(0,z)$ for all $g \in A$ so that the first sum cannot exceed

$$\exp -d_{\mathbb{D}}(0,z)|A| \leq 2|A|(1-|z|^2)$$

where $|A|$ is the cardinality of A.

Now we need to show that here is a $\delta > 0$ and $0 < R < 1$ so that $|A|$ is bounded above by a constant dependent only on R and Γ. To see this note that the free edge of $D(0)$ is properly contained in an interval of discontinuity $\sigma \subset \Omega(\Gamma) \cap \partial\mathbb{D}$. Let $\epsilon > 0$ and define N_ϵ to be an ϵ-neighbourhood of I. Then we may chose ϵ small enough so that N_ϵ is contained in $\mathbb{D} \cup \sigma \cup \mathbb{D}^*$ where $\mathbb{D}^*$ denotes the exterior of $\mathbb{D}$. By the discontinuity of Γ on $\Omega(\Gamma)$ we may deduce that there are only finitely many $g \in \Gamma$ with $g(0) \in \overline{N_\epsilon}$. Therefore if we choose $\delta > 0$ small enough then $|A| < K$ for some constant K that only depends on Γ and R.

Hence the first sum on the left hand side of (9) is bounded by

$$2K(1-|z|^2).$$

To complete the proof of the result note that since

$$\frac{1-|g(z)|^2}{1-|z|^2} = \frac{1-|g^{-1}(0)|^2}{|1-\overline{g^{-1}(0)}z|^2},$$

the second sum on the right of (9) is bounded by

$$\begin{aligned} 2(1-|z|^2)\sum_{g\in B}\frac{1-|g(z)|^2}{1-|z|^2} &= 2(1-|z|^2)\sum_{g\in B}\frac{1-|g^{-1}(0)|^2}{|1-\overline{g^{-1}(0)}z|^2} \\ &\leq 2(1-|z|^2)\sum_{g\in B}\frac{1-|g^{-1}(0)|^2}{|z-g^{-1}(0)|^2} \\ &\leq (1-|z|^2)\frac{2\delta^{-1}}{\mathrm{dist}(z,\Lambda(\Gamma))^2}\rho_\Gamma(0,0;1). \end{aligned}$$

Now the free edge of $D(0)$ is properly contained in $\Omega(\Gamma)\cap\partial\mathbb{D}$. To see this it follows from [3, Theorem 10.2.3] that if either of the end-points of the free-edge of $D(0)$ are limit points of Γ then it is a parabolic fixed point, and this cannot be so since it is not a cusp in $\mathcal{F}$. Therefore $\mathrm{dist}(z,\Lambda(\Gamma))$ is bounded below by some fixed constant dependent on $D(0)$ whenever $|z|$ is close enough to 1.

Collecting estimates of (9) together we obtain, for $z \in D(0)$,

$$\begin{aligned} \rho_\Gamma(0,z;1) &\leq \left(2K + \frac{2\delta^{-1}}{\mathrm{dist}(z,\Lambda(\Gamma))^2}\right)(1-|z|^2) \\ &\leq C\frac{1-|z|}{1+|z|} = C\exp -d_{\mathbb{D}}(0,z) \end{aligned}$$

where C is a constant that depends on Γ and $D(0)$. Since $z \in D(0)$ the result is proved. □

We also need the following result classifying limit points for finitely generated Fuchsian groups, see [4, Theorem 10.2.5].

Lemma 2. Γ *is finitely generated if and only if each* $\zeta \in \Lambda(\Gamma)$ *is either*

1. *a fixed point for a parabolic element of* Γ*; or*

2. *a point of approximation – i.e. there is a sequence g_n, $n = 1, 2, \ldots$, of elements of Γ such that $g_n(0) \to \zeta$ non-tangentially.*

Proof of Theorem 2. Let $\zeta \in \partial\mathbb{D}$ be arbitrary. If ζ is a parabolic fixed point, then $\phi(z) \to p_j$ for some j when $z \to \zeta$, see [8, Corrolary 1, p216]. Since $|p_j| < 1$ it follows that ϕ has infinite angular derivative there.

Similarly if ζ is a point of approximation then, with g_n a suitable sequence such that $g_n(0) \to \zeta$ as $n \to \infty$, we have that

$$|\phi(g_n(0))| = |\phi(g_0(0))| < 1$$

and since $g_n(0)$ converges non-tangentially, it follows from the Julia-Caratheodory theorem ([5, Theorem 2.44]) that the angular derivative at ζ is

$$|\phi'(\zeta)| = \lim_{n\to\infty} \frac{1 - |\phi(g_n(0))|}{1 - |z|} = \lim_{n\to\infty} \frac{1 - |\phi(g_0(0))|}{1 - |z|} = \infty.$$

From Lemma 2 all other points in $\partial\mathbb{D}$ are in the complement of the limit set of Γ.

Suppose first that ϕ has infinite angular derivative at all points $\zeta \in \partial\mathbb{D}\backslash\Lambda(\Gamma)$. Then from Lemma 1

$$\lim_{z\to\zeta} \frac{1 - |\phi(z)|}{\rho_\Gamma(0, z; 1)} \geq c \lim_{z\to\zeta} \frac{1 - |\phi(z)|}{1 - |z|} = \infty.$$

The compactness of C_ϕ now follows from Theorem 1.

Suppose, conversely, that C_ϕ is compact. Let $\zeta \in \partial\mathbb{D}\backslash\Lambda(\Gamma)$ and $I \subset \partial D(0)$ be the free edge of $D(0)$. There is a $h \in \Gamma$ such that $\zeta \in h(I)$ and we may suppose without loss of generality that $z \to \zeta$ inside

$$h(D(0)) = D(h(0)).$$

Then by continuity of h^{-1}, as $z \to \zeta$,

$$z^* = h^{-1}(z) \to h^{-1}(\zeta) = \zeta^*$$

and $\zeta^* \in I$.

From Lemma 1 we thus have

$$\begin{aligned}
\liminf_{z\to\zeta} \frac{1 - |\phi(z)|}{1 - |z|} &= \liminf_{z^*\to\zeta^*} \frac{1 - |\phi(z^*)|}{1 - |z^*|} \cdot \frac{1 - |z^*|}{1 - |z|} \\
&= \frac{1}{|h'(\zeta)|} \liminf_{z^*\to\zeta^*} \frac{1 - |\phi(z^*)|}{1 - |z^*|} \\
&\geq c \liminf_{z^*\to\zeta^*} \frac{1 - |\phi(z^*)|}{\rho_\Gamma(0, z^*; 1)} = \infty.
\end{aligned}$$

Therefore ϕ has infinite angular derivative at each $\zeta \in \partial\mathbb{D}$ as required. □

Theorem 3 follows almost immediately from Theorem 2 given certain properties of inner functions. First, for a given inner function I, a *singular point* is a point $\eta \in \partial\mathbb{D}$ such that I cannot be extended to be analytic in a neighbourhood of η. The set of singular points of a universal cover of $\mathbb{D}$ onto $\mathbb{D}\backslash\{p_1, \ldots, p_n\}$ is easily seen to be the limit set of the uniformizing Fuchsian group. In fact if $\eta \in \Lambda(\Gamma)$ then in each neighborhood there are infinitely many zeros of $I - a$ for any $a \neq p_i$, $i = 1, \ldots, n$ and so η is singular. To prove the contrapositive recall that Γ acts discontinuously on the larger set $\Omega(\Gamma)$. Therefore I may be extended to a holomorphic function on $\Omega(\Gamma)$ by considering the universal covering map of $\Omega(\Gamma)$ onto the so-called Schottky double $\Omega(\Gamma)/\Gamma$. It follows that if $\eta \notin \Lambda(\Gamma)$ then η is not singular. Note in this case I' exists in the normal sense on $\partial\mathbb{D}\backslash\Lambda(\Gamma)$ and is non-zero there since it is conformal, furthermore since I is inner the absolute value of $I'(\eta)$ coincides with the angular derivative.

Proof of Theorem 3. The proof of this result follows from the properties above and the Julia-Caratheodory theorem, we will merely sketch the details here.

First note that

$$\psi^{-1} \circ \phi\colon \mathbb{D} \to \mathbb{D}\backslash\{p_1', \ldots, p_n'\}$$

for $p_i' = \psi(p_i)$, $i = 1, \ldots, n$. It follows from uniqueness that $I = \psi^{-1} \circ \phi$ is the universal covering map of $\mathbb{D}$ onto $\mathbb{D}\backslash\{p_1', \ldots, p_n'\}$. Clearly it is also an inner function so that the remarks above apply.

Suppose first that C_ψ is compact, then

$$C_\phi = C_I C_\psi. \tag{10}$$

Since the compact operators form a left ideal in the algebra of bounded operators this means that C_ϕ is compact.

To prove the converse, suppose C_ϕ is compact. Then ϕ has infinite angular derivative at all points of $\partial\mathbb{D}$ by Theorem 2. Suppose now that $\eta \in \partial\mathbb{D}$ is a point at which $|\psi'(\eta)| < \infty$.

Let $\mathcal{F}$ be locally finite fundamental domain for the uniformizing group Γ of $\mathbb{D}\backslash\{p_1', \cdots, p_n\}$, then we may find $\zeta \in \partial\mathcal{F}$ such that

$$\lim_{r\to 1} I(r\zeta) = \eta,$$

furthermore ζ is in the free side of $\mathcal{F}$ and hence, from the remarks above, that $|I'(\zeta)| < \infty$ and that the limit above is non-tangential. It follows that

$$\begin{aligned} \lim_{r\to 1} \phi'(r\zeta) &= \lim_{r\to 1} \psi'(I(r\zeta)) \cdot I'(r\zeta) \\ &= \lambda\psi'(\eta)|I'(\eta)| \end{aligned}$$

for some $|\lambda| = 1$. Since the right hand side above is finite we have a contradiction and hence $|\psi'(\eta)| = \infty$.

□

Remarks: The proof of Theorem 2 shows that, in fact, C_ϕ is compact if and only if ϕ has infinite angular derivative at all points in the complement, in $\partial\mathbb{D}$, of the limit set since the non-existence of the angular derivative on the limit set depends only on the fact that Γ is finitely generated.

Note also that one half of the assertion of Theorem 2 follows from the result of Shapiro and Taylor's mentioned earlier, see [12] or [5, Corollary 3.14], therefore the argument above gives a new proof of this result when ϕ is a universal covering map.

5 Concluding Remarks

In this paper we have utilized a link between the Nevanlinna counting function of a universal covering map and the Poincare series of its underlying Fuchsian group. This relationship can potentially be exploited throughout the study of composition operators to generalise results concerning univalent conformal maps to universal covering maps since the Nevanlinna counting function and variations thereof occur naturally throughout the study of composition operators. For example C_ϕ is in the Schatten p-class of H^2 if, and only if,

$$\iint_{\mathbb{D}} \left(\frac{\mathcal{N}_\phi(w)}{\log 1/|w|} \right)^{p/2} d\lambda < \infty$$

where $d\lambda = (1-|w|)^{-2} dA$, and A is Lebesgue 2-dimensional measure on $\mathbb{D}$.

We have studied the case of finitely connected domains in this work and this constraint pervades all the proofs. It is not clear how much of the work in this paper, if any, could be extended to the case of infinitely connected domains.Of note is the work of [1] where the authors construct a universal covering map, B, of $\mathbb{D}$ onto $\mathbb{D}\backslash S$, where S is a discrete set such that, for all $z \in \mathbb{D}$

$$(1-|z|^2)|B'(z)| \leq w(1-|B(z)|^2)$$

for a given continuous function $w\colon (0,1] \to (0,\infty)$.

Fuchsian groups for which the Poincare series of order 1 converge are said to be of *convergence type*. Obviously this is necessary for the results in this paper to be extended to infinitely connected domains and there is considerable work done on understanding convergence and divergence type Fuchsian groups.

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