



The differential invariants of $SL_2(\#_3)$ acting on trace-free matrices over $\#_3$

Jonathan Elmer & Anja Meyer

To cite this article: Jonathan Elmer & Anja Meyer (12 May 2026): The differential invariants of $SL_2(\mathbb{F}_3)$ acting on trace-free matrices over $\mathbb{F}_3$, Communications in Algebra, DOI: 10.1080/00927872.2026.2660143

To link to this article: <https://doi.org/10.1080/00927872.2026.2660143>



© 2026 The Author(s). Published with license by Taylor & Francis Group, LLC.



Published online: 12 May 2026.



Submit your article to this journal [↗](#)



Article views: 34



View related articles [↗](#)



View Crossmark data [↗](#)

The differential invariants of $SL_2(\mathbb{F}_3)$ acting on trace-free matrices over $\mathbb{F}_3$.

Jonathan Elmer^a and Anja Meyer^b

^aMiddlesex University, London, UK; ^bUniversity of Manchester, Manchester, UK

ABSTRACT

Let M denote the vector space of 2×2 matrices with coefficients in $\mathbb{F}_3$ and trace zero. Let $G = SL_2(\mathbb{F}_3)$. Then G acts on M via conjugation. Let $R = (S(M^*) \otimes \Lambda(M^*))$ be the algebra of differential forms on M . We compute a minimal generating set for R^G as a commutative-graded algebra. In doing so we utilize the theory of Cohen-Macaulay modules and results in the theory of covariants.

ARTICLE HISTORY

Received 12 December 2025
Communicated by Ellen Kirkman

KEYWORDS

Characteristic p ;
cohomology; finite groups;
invariant theory

2020 MATHEMATICS

SUBJECT CLASSIFICATION

13A50; 14L24

1. Introduction

Computing invariant rings can be a challenging task, especially in dividing characteristic. The theory of Cohen-Macaulay modules (e.g., Chapter 2.8 in [5]) can offer some useful theory to use. Invariant theory plays an important role in group cohomology computations, and we are keen to find new methods for computations. As a first fact we recall that for any finite group G and field k , $H^0(G, k) = k^G$. We also observe for a normal subgroup N of G , the action of G/N on N induces a linear action of G/N on $H^*(N, k)$. Moving to modular coefficients and using the theory of stable elements, by Chapter II.6 in [1] $H^*(G, \mathbb{F}_p) \cong H^*(N, \mathbb{F}_p)^{G/N}$. Note that if $N = P$ is further a Sylow p -subgroup, this is an application of non-modular invariant theory, as in that case $|G/P|$ is coprime to $|G|$. But even for not necessarily normal subgroups of G , invariant theory plays a role: a result of Swan [12] states that for any abelian Sylow p -subgroup P of G the composition $H^*(P, \mathbb{F}_p)^{N_G(P)} \cong \text{im}(\text{res}_P^G) \cong H^*(G, \mathbb{F}_p)$, where the second isomorphism is due to the theory of stable elements as seen in [4]. A more specific example is found in [10] where the cohomology ring $H^*(SL_2(\mathbb{Z}/3^n), \mathbb{F}_3)$, for $n > 1$ is computed via stable elements, where the invariant ring $(S(M^*) \otimes \Lambda(M^*))^{SL_2(\mathbb{F}_3)}$, for M a module of dimension 3, is a main ingredient. This is the invariant ring described in the abstract. In this paper, we compute a minimal set of generators of this graded-commutative invariant ring explicitly, using a combination of Cohen-Macaulay modules and covariants.

We start by giving the necessary background material in modular and non-modular invariant theory, as the computations in this paper combine both: on the one hand, the characteristic of $\mathbb{F}_3$ divides $|SL_2(\mathbb{F}_3)|$; on the other hand, we will show that M is isomorphic to a module induced from a non-modular subgroup of $SL_2(\mathbb{F}_3)$. Then, using a minor generalization of [7, Proposition 5] (proved here as Lemma 2) moves us into the non-modular case. This trick enables the use of tools from the non-modular setting, such as the Molien series.

CONTACT Jonathan Elmer  j.elmer@mdx.ac.uk  Middlesex University, London, UK.

© 2026 The Author(s). Published with license by Taylor & Francis Group, LLC.

This is an Open Access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. The terms on which this article has been published allow the posting of the Accepted Manuscript in a repository by the author(s) or with their consent.

The background section is followed by an explicit computation of the generators of $(S(M^*) \otimes \Lambda(M^*)^{\text{SL}_2(\mathbb{F}_3)})$. These are given in [Theorem 9](#), which is the main theorem of this paper.

2. Background

Let k be a field, G a finite group and V a finite-dimensional kG -module. Let $v_1, v_2, \dots, v_n$ be a basis of V and let $x_1, x_2, \dots, x_n$ be the corresponding dual basis of V^* . The action of G on V induces an action of G on V^* defined by

$$(g(f))(v) = f(g^{-1}(v))$$

and we may extend this, by algebra automorphisms, to an action on the k -algebra $k[V] := S(V^*)$. Setting $\deg(x_i) = 2$ for each i endows $S(V^*)$ with a grading which is respected by the G -action. The fixed points $k[V]^G$ of this action form a graded subalgebra called the subalgebra of invariants.

Now let W be a further kG -module with basis $w_1, w_2, \dots, w_m$. Let $k[V] \otimes W$ denote the free $k[V]$ -module spanned by this basis. G acts on $k[V] \otimes W$ via the formula

$$g(f \otimes w_i) = g(f) \otimes g(w_i)$$

for $f \in k[V]$, extended linearly, and one may now show that the fixed points $(k[V] \otimes W)^G$ of this action form a $k[V]^G$ -module. This is called the module of W -covariants. Since $k[V]$ is finitely generated over $k[V]^G$, the same is true of $(k[V] \otimes W)$. Since $k[V]^G$ is Noetherian and $(k[V] \otimes W)^G$ is a submodule of $k[V] \otimes W$, we get that the module W -covariants is finitely generated in general.

Consider the special case $R_i(V) := k[V] \otimes (\Lambda^i(V^*))$. This is called the module of differential i -forms. Usually one writes $dx_1, dx_2, \dots, dx_n$ for the $k[V]$ -basis of $R_i(V)$, which explains the terminology. The exterior product $\wedge : \Lambda^i(V^*) \times \Lambda^j(V^*) \rightarrow \Lambda^{i+j}(V^*)$ can be extended $k[V]$ -linearly to an exterior product on $R(V) := \bigoplus_{i=0}^n R_i(V) = S(V^*) \otimes \Lambda(V^*)$ (note that $\Lambda^i(V) = 0$ for $i > n$) which makes $R(V)$ into a non-commutative k -algebra. The fixed points $R(V)^G$ are called the algebra of differential invariants.

Another way of viewing $R(V)$ is as follows: write $y_i := dx_i$ for each i and endow $\Lambda^*(V^*) = \Lambda[y_1, y_2, \dots, y_n]$ with a grading by setting $\deg(y_i) = 1$ for each i . Then $R(V)$ is a commutative-graded k -algebra, i.e. for each $f, g \in R$ we have

$$fg = (-1)^{\deg(f)\deg(g)}gf.$$

The goal of this paper is to find a minimal generating set for $R(M)^G$, where M is the vector space of 2×2 traceless matrices over $\mathbb{F}_3$ and the action of $G = \text{SL}_2(F_3)$ is by conjugation. In order to determine the module generators of the summands $R_i(M)$, it is helpful to recall the following notions.

For S a finitely generated k -algebra, k a field, and B a finitely generated S -module, a homogeneous system of parameters of positive degree for B is a set of elements $\{f_1, \dots, f_n\} \subset S$ where n is the Krull-dimension of B , with the property that B is finitely generated as a module over the ring $A = k[f_1, \dots, f_n]$. It can be shown that if B a free module over a subalgebra of S generated by a homogeneous system of parameters, it is free over any such subalgebra. We say B is a Cohen-Macaulay S -module if it has this property. In addition, S is a Cohen-Macaulay algebra if it is a Cohen-Macaulay module over itself.

A further fundamental concept used in this paper is a version of Molien's formula for the Hilbert series of a ring of relative invariants. For S a finitely generated k -algebra with homogeneous components S_d , the Hilbert series of R is the power series

$$H(S, t) = \sum_{0 \leq i} \dim_k(S_i) t^i. \quad (1)$$

Let G be a finite group, V a vector space and k a field with $\text{char}(k) = 0$. Further let $\chi : G \rightarrow k$ be a linear character. We say $f \in k[V]$ is a χ -relative invariant if $g(f) = \chi(g)f$ for all $g \in G$. The set of χ -relative G -invariants is denoted $k[V]_\chi^G$ and is a module over $k[V]^G$. Molien's formula (found for example in [5, Theorem 3.7.1]) was established for k such that $\text{char}(k) = 0$. In fact we will need a version of this

result in a finite field of order p not dividing $|G|$. We proceed as follows: let e be the exponent of G . Let K be a field of characteristic 0 containing a primitive e th root of unity and choose a homomorphism $\phi : k \rightarrow K$ which identifies k with the multiplicative group of e th roots of unity in K . For any linear character $\chi : G \rightarrow k$ we define the Brauer lift $\chi^0 : G \rightarrow K$ by $\chi^0 = \phi \circ \chi$. Further, for any matrix $X \in \text{GL}_n(k)$ we define $\det^0(X) \in K$ to be the product of $\phi(\lambda)$ as λ runs through the eigenvalues of X . The following formula is Theorem 2.5.2 in [3]

$$H(k[V]_\chi^G, t) = \frac{1}{|G|} \left(\sum_{g \in G} \frac{\chi(g)}{\det(\text{Id}_V - tg)} \right), \quad (2)$$

and it holds for k a field of characteristic zero. As observed in [3], the formula holds when k is a field of finite characteristic not dividing $|G|$ after replacing χ and $\det$ by χ^0 and $\det^0$, respectively.

The computation of the generators of the summands in $R_i(M)^G$ relies on some material in the study of covariants. In particular, we will require the following fundamental result, due to Eagon and Hochster [8]

Lemma 1. *Let k be a field whose characteristic does not divide $|G|$. Then $(k[V] \otimes W)^G$ is a Cohen-Macaulay $k[V]^G$ -module.*

Let $H \leq G$, T be a left-transversal of H in G , and $\{e_t | t \in T\}$ be a basis for the left permutation module $k[G/H]$. Recall that H acts trivially on this. We define a map

$$\begin{aligned} \Theta : k[V]^H &\rightarrow k[V] \otimes k[G/H] \\ \Theta : f &\mapsto \sum_{t \in T} t f \otimes e_t. \end{aligned}$$

The following result on covariants is a minor generalization of Proposition 5 in [7] and its proof is crucial for our computation:

Lemma 2. *Let k be a field, G a group, and $\mathcal{A}$ a kG -algebra. Let $H \leq G$ and let T be a set of left coset representatives for H in G . Then the map $\Theta : \mathcal{A}^H \rightarrow (\mathcal{A} \otimes k[G/H])^G$ defined by*

$$\Theta(f) = \sum_t t(f \otimes t)$$

is an isomorphism of $\mathcal{A}^G$ -modules.

Proof. An adaptation of the proof of [7, Proposition 5] with $\mathcal{A}$ replacing $k[V]$ throughout. □

We shall continue with the computation of $(S(M^*) \otimes \Lambda(M^*))^{SL_2(\mathbb{F}_3)}$.

3. Computing invariants

It is convenient, for reasons which will become clear later, to choose a basis

$$v_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

for M . Let x_1, x_2, x_3 be the corresponding dual basis for M^* , which will generate $S(M^*)$ as a commutative algebra. Let y_1, y_2, y_3 be the dual basis of generators for $\Lambda(M^*)$. Now R can be viewed as a free graded-commutative algebra generated over $\mathbb{F}_3$ by $x_1, x_2, x_3, y_1, y_2, y_3$ in which $\deg(x_i) = 2$ and $\deg(y_i) = 1$. We define an action of G on M by setting

$$g(v) = gv g^{-1}$$

for $g \in G$ and $v \in M$. The induced action on M^* is then denoted by $g(f)$ for $g \in G$ and $f \in M^*$, and this extends algebraically to an action on R .

We label certain elements of G : let

$$t := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which generates the Sylow-3-subgroup $P \cong C_3$ of G . Let

$$i := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that t and i together generate G . Further, the kernel K of the action of G on M is $\{\pm e\} = \{e, i^2\}$ where e denotes the identity matrix. Further, we set

$$j := t^{-1}it; k := tit^{-1}.$$

Notice that $i^2 = j^2 = k^2 = -e$ and $ij = k, jk = i, ki = j$; in other words, i, j generate a subgroup $H = \{\pm e, \pm i, \pm j, \pm k\} \cong Q_8$. Moreover, the matrices representing j and k are the same as v_3 and v_2 respectively, from which we see immediately that

$$t(v_s) = v_{s+1}$$

with subscripts understood modulo 3. Let $\rho : G \rightarrow \text{GL}_3(\mathbb{F}_3)$ be the representation afforded by the action of G on M . Then we have

$$\rho(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\rho(i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Each of these is their own inverse-transpose, therefore

$$t(x_s) = x_{s+1},$$

ditto y_s . Similarly one may check that

$$i(x_1) = x_1, i(x_2) = -x_2, i(x_3) = -x_3,$$

ditto y_s again. For future use we note that

$$j(x_1) = -x_1, j(x_2) = -x_2, i(x_3) = x_3,$$

and

$$k(x_1) = -x_1, k(x_2) = x_2, i(x_3) = -x_3.$$

Define a one-dimensional representation $W = \langle w \rangle$ of H as follows:

$$i(w) = w, j(w) = -w, k(w) = -w.$$

Let χ denote the corresponding character $H \rightarrow \mathbb{F}_3^\times$. The following observation is crucial:

Lemma 3. *We have $M^* \cong \mathbb{F}_3(G/H) \otimes W$.*

Proof. $\{e, t, t^2\}$ is a set of left coset representatives for H in G . Define $\phi : M^* \rightarrow \mathbb{F}_3(G/H) \otimes W$ as follows:

$$\phi(x_s) = t^{s-1} \otimes w.$$

Then

$$t(\phi(x_s)) = t^s \otimes w = \phi(x_{s+1}) = \phi(t(x_s))$$

and

$$\begin{aligned}
 i(\phi(x_1)) &= i((e \otimes w)) &= e \otimes (i(w)) &= e \otimes w &= \phi(x_1) = \phi(i(x_1)); \\
 i(\phi(x_2)) &= i((t \otimes w)) &= t \otimes (j(w)) &= -t \otimes w &= -\phi(x_2) = \phi(i(x_2)); \\
 i(\phi(x_3)) &= i((t^2 \otimes w)) &= t^2 \otimes (k(w)) &= -t^2 \otimes w &= -\phi(x_3) = \phi(i(x_3)).
 \end{aligned}$$

So ϕ is an isomorphism. □

The ring of invariants $S(M_p^*)^{G_p}$, where M_p denotes the 2×2 trace-free matrices over $\mathbb{F}_p$ and $G_p = \text{SL}_2(\mathbb{F}_p)$, was computed by Anghel [2]. Translating his result for $p = 3$ into our preferred basis, we obtain:

Proposition 4. *Let*

$$\begin{aligned}
 a_1 &= x_1^2 + x_2^2 + x_3^2; \\
 a_2 &= x_1 x_2 x_3 \\
 a_3 &= x_1^4 + x_2^4 + x_3^4
 \end{aligned}$$

The algebra of invariants $S(M^)^G$ is Cohen-Macaulay, and a free module over $A := K[a_1, a_2, a_3]$ generated by 1 and*

$$b := x_1^4 x_2^2 + x_1^2 x_3^4 + x_2^4 x_3^2.$$

Remark 5. As $\dim(M^P) = 1 \geq \dim(M) - 2$ we get that $S(M^*)^G$ is Cohen-Macaulay by [9]. It is easy to see directly that $\{a_1, a_2, a_3\}$ form a homogeneous system of parameters. Now by Theorem 3.7.1 in [6] we get that $S(M^*)^G$ has rank 2 over A (remember that the action of G is not faithful, it is really an action of G/K which has order 12). We see that $\{1, b\}$ are the secondary invariants by computing the invariants in all degrees up to six.

We recall that Λ is a graded algebra consisting of vector spaces and as such we can decompose $R = S(M^*) \otimes \Lambda(M^*)$ as

$$R = \bigoplus_{i=0}^3 S(x_1, x_2, x_3) \otimes \Lambda^i(y_1, y_2, y_3).$$

We shall compute the invariants for each of these summands individually.

The actions of G on $\Lambda^3(M^*) = \langle y_1 y_2 y_3 \rangle$ and $\Lambda^0(M^*) = \langle 1 \rangle$ are trivial. We obtain immediately:

Proposition 6.

- (a) $(S(M^*) \otimes \Lambda^3(M^*))^G$ is a free A -module generated by $y_1 y_2 y_3$ and $b y_1 y_2 y_3$;
- (b) $(S(M^*) \otimes \Lambda^0(M^*))^G$ is a free A -module generated by 1 and b .

On the other hand we have

$$\Lambda^1(M^*) = \langle y_1, y_2, y_3 \rangle \cong M^* \text{ and } \Lambda^2(M^*) = \langle y_2 y_3, y_3 y_1, y_1 y_2 \rangle \cong M^*.$$

Applying Lemma 2 to the above with $H = Q_8$, $G = \text{SL}_2(F)$ and $\mathcal{A} = S(M^*) \otimes W$ yields a pair of $S(M^*)^G$ -module isomorphisms $\Theta_i : (S(M^*) \otimes W)^H \rightarrow (S(M^*) \otimes \Lambda^i(M^*))^G$ for $i = 1, 2$, defined by

$$\begin{aligned}
 \Theta_1(f) &= \sum_{j=0}^2 t^j (f \otimes y_{j+1}) \\
 \Theta_2(f) &= \sum_{j=0}^2 t^j (f \otimes y_{j+2} y_{j+3})
 \end{aligned}$$

with indices understood modulo 3.

Note that elements of $(S(M^*) \otimes W)^H$ are precisely those $f \in S(M^*)$ satisfying $h(f) = \chi(h)f$. In other words, they are the relative invariants $S(M^*)_\chi^H$. As $|H| \not\equiv 0 \pmod 3$, $S(M^*)_\chi^H$ is a Cohen-Macaulay $S(M^*)^H$ -module, by [Lemma 1](#). It cannot be a free $S(M^*)^H$ -module, by [11], since H is not a reflection group. However, since $(S(M^*) \otimes W)^H$ is a finitely generated $S(M^*)^H$ module, which is in turn a finitely generated A -module, $(S(M^*) \otimes W)^H$ is finitely generated over A . Since it is Cohen-Macaulay, it is therefore a free A -module. We have therefore proved that $S(M^*) \otimes \Lambda^i(M^*)^G$ is a free A -module, for $i = 1, 2$. In particular, this shows that $R(M)$ is a Cohen-Macaulay $S(M^*)^G$ -module.

It remains to find generators for $S(M^*)_\chi^H$. Note once more that the action of H is not faithful as $K \leq H$. Write $\bar{H} = H/K$. As $\bar{H}$ is non-modular we can use our adaptation of (2) to compute the Hilbert Series of $S(M^*)_\chi^H$. We find:

$$\begin{aligned} H(S(M^*)_\chi^H, t) &= \frac{1}{4} \sum_{\sigma \in \bar{H}} \frac{\chi^0(\sigma)}{\det^0(\text{id}_M - \rho(\sigma)t)} \\ &= \frac{1}{4} \left(\frac{1}{(1-t)^3} + \frac{1}{(1-t)(1+t)^2} + 2 \frac{-1}{(1-t)(1+t)^2} \right) \\ &= \frac{1}{4} \left(\frac{1}{(1-t)^3} - \frac{1}{(1-t)(1+t)^2} \right) \\ &= \frac{1}{4} \left(\frac{(1+t)^2 - (1-t)^2}{(1-t)^3(1+t)^2} \right) \\ &= \frac{t}{(1-t)^3(1+t)^2} \\ &= \frac{t}{(1-t)(1-t^2)^2} \\ &= \frac{t+t^2}{(1-t^2)^3} \end{aligned}$$

A homogeneous system of parameters for $S(M^*)^H$ is given by $\{x_1^2, x_2^2, x_3^2\}$, and the above shows that $S(M^*)_\chi^H$ is a free module over the subalgebra $F[x_1^2, x_2^2, x_3^2]$ with generators in degrees 1 and 2, which are easily seen to be x_1 and x_2x_3 . However, we need generators over A , so we rewrite the series as

$$\begin{aligned} \frac{(t+t^2)(1+t^2)}{(1-t^2)(1-t^2)(1-t^4)} &= \frac{(t+t^3)}{(1-t^2)(1-t)(1-t^4)} \\ &= \frac{(t+t^3)(1+t+t^2)}{(1-t^2)(1-t^3)(1-t^4)} \\ &= \frac{(t+t^2+2t^3+t^4+t^5)}{(1-t^2)(1-t^3)(1-t^4)}. \end{aligned}$$

Thus, $S(M^*)_\chi^H$ is a free A -module with generators in degrees 1,2,3 (twice), 4 and 5.

Proposition 7. $S(M^*)_\chi^H$ is generated freely over A by the set

$$\{x_1, x_2x_3, x_1^3, x_1x_2^2, x_2^3x_3, x_1^3x_2^2\}.$$

Proof. It is easy to see each of the given monomials lies in $S(M^*)_\chi^H$. Let B be the A -module generated by the given set. By the previous paragraph, the elements of $S(M^*)_\chi^H$ are sums of monomials of the form $m = x_1^i x_2^j x_3^k$ where $i \not\equiv j \equiv k \pmod 2$. We will prove by induction on $d := \deg(m)$ that every such monomial belongs to B .

If $d = 1$, then $m = x_1$ and we are done. If $d = 2$, then $m = x_2x_3$ and we are done.

If $d = 3$ then $m = x_1^3, m = x_1x_2^2$ or $m = x_1x_3^2$. In the first two cases we are done, and in the third we have

$$x_1x_3^2 = a_1x_1 - x_1^3 - x_1x_2^2 \in B$$

as desired.

If $d = 4$ then $m = x_1^2x_2x_3, m = x_2^3x_3$ or $m = x_2x_3^2$. In the first case we have

$$m = a_2x_1 \in B.$$

In the second case we are done, and in the third we have

$$x_2x_3^2 = a_1x_2x_3 - a_2x_1 - x_2^3x_3 \in B$$

as desired.

If $d = 5$ then $m \in \{x_1^5, x_1^3x_2^2, x_1^3x_3^2, x_1x_2^4, x_1x_2^2x_3^2, x_1x_3^4\}$. We have $x_1^3x_2^2 \in B$ and $x_1x_2^2x_3^2 = a_2x_2x_3 \in B$. Now we have

$$a_1x_1x_2^2 = x_1^3x_2^2 + x_1x_2^4 + x_1x_2^2x_3^2$$

which shows that $x_1x_2^4 \in B$. Further,

$$a_1^2x_1 - a_3x_1 = -x_1^3x_2^2 - x_1^3x_3^2 - x_1x_2^2x_3^2$$

which shows that $x_1^3x_3^2 \in B$. Now

$$a_1x_1^3 = x_1^5 + x_1^3x_2^2 + x_1^3x_3^2$$

which shows that $x_1^5 \in B$. And finally,

$$a_3x_1 = x_1^5 + x_1x_2^4 + x_1x_3^4$$

which shows that $x_1x_3^4 \in B$ as required.

Now assume that $d \geq 6$ and assume the claim is proven for all monomials of degree $< d$. Suppose first that d is even, then we have $m = x_1^{2i}x_2^{2j+1}x_3^{2k+1}$ for some $i, j, k \geq 0$ with $i + j + k \geq 2$. If $i \geq 1$ we may write

$$m = a_2x_1^{2i-1}x_2^{2j}x_3^{2k} \in B$$

by induction, so we may assume $i = 0$ and $j + k \geq 2$. Let $j \geq 1$, then we have

$$\begin{aligned} m &= a_1x_2^{2j-1}x_3^{2k+1} - x_1^2x_2^{2j-1}x_3^{2k+1} - x_2^{2j-1}x_3^{2k+3} \\ &= a_1x_2^{2j-1}x_3^{2k+1} - a_2x_1x_2^{2j-2}x_3^{2k} - x_2^{2j-1}x_3^{2k+3} \end{aligned}$$

By induction, the first two terms lie in B and we have

$$x_2^{2j+1}x_3^{2k+1} = -x_2^{2j-1}x_3^{2k+3} \pmod{B} \quad (3)$$

for all $j \geq 1$. Iterating the argument shows that

$$x_2^{2j+1}x_3^{2k+1} = x_2^{2j-3}x_3^{2k+5} \pmod{B} \quad (4)$$

for all $j \geq 2$. However, we also have

$$\begin{aligned} m &= a_3x_2^{2j-3}x_3^{2k+1} - x_1^4x_2^{2j-3}x_3^{2k+1} - x_2^{2j-3}x_3^{2k+5} \\ &= a_3x_2^{2j-3}x_3^{2k+1} - a_2x_1^3x_2^{2j-4}x_3^{2k} - x_2^{2j-3}x_3^{2k+5} \end{aligned}$$

which shows that

$$x_2^{2j+1}x_3^{2k+1} = -x_2^{2j-3}x_3^{2k+5} \pmod{B} \quad (5)$$

for all $j \geq 2$. By (3) and (4) we must have

$$x_2^{2j-3}x_3^{2k+5} = x_2^{2j-1}x_3^{2k+3} = x_2^{2j+1}x_3^{2k+1} = 0 \pmod{B}$$

for all $j \geq 2, k \geq 0$ as required.

Now suppose that d is odd, so $d \geq 7$ and $m = x_1^{2i+1}x_2^{2j}x_3^{2k}$ for some $i, j, k \geq 0$ with $i + j + k \geq 3$. If $j \geq 1$ and $k \geq 1$ then we may write

$$m = a_2 x_1^{2i} x_2^{2j-1} x_3^{2k-1} \in B$$

by induction. So we may assume $j = 0$ or $k = 0$. Assume $k = 0$, then $i + j \geq 3$, and we have, for all $i, j \geq 1$,

$$\begin{aligned} m &= a_1 x_1^{2i-1} x_2^{2j} - x_1^{2i-1} x_2^{2j+2} - x_1^{2i-1} x_2^{2j} x_3^2 \\ &= a_1 x_1^{2i-1} x_2^{2j} - x_1^{2i-1} x_2^{2j+2} - a_2 x_1^{2i-2} x_2^{2j-1} x_3 \end{aligned}$$

By induction, the first and last terms lie in B and we have

$$x_1^{2i+1} x_2^{2j} = -x_1^{2i-1} x_2^{2j+2} \pmod{B} \quad (6)$$

for all $i, j \geq 1$. Iterating the argument shows that

$$x_1^{2i+1} x_2^{2j} = x_1^{2j-3} x_2^{2j+4} \pmod{B} \quad (7)$$

for all $i \geq 2, j \geq 1$. However, we also have

$$\begin{aligned} m &= a_3 x_1^{2i-3} x_2^{2j} - x_1^{2i-3} x_2^{2j+4} - x_1^{2i-3} x_2^{2j} x_3^2 \\ &= a_1 x_1^{2i-3} x_2^{2j} - x_1^{2i-3} x_2^{2j+4} - a_2 x_1^{2i-4} x_2^{2j-1} x_3 \end{aligned}$$

for all $i \geq 2, j \geq 1$ which shows that

$$x_1^{2i+1} x_2^{2j} = -x_1^{2i-3} x_2^{2j+4} \pmod{B}. \quad (8)$$

By (6) and (7) we must have

$$x_1^{2i+1} x_2^{2j} = x_1^{2i-1} x_2^{2j+2} = x_1^{2i-3} x_2^{2j+4} = 0 \pmod{B} \quad (9)$$

for all $i \geq 2$ and $j \geq 1$. A similar argument shows that

$$x_1^{2i+1} x_3^{2k} = x_1^{2i-1} x_3^{2k+2} = x_1^{2i-3} x_3^{2k+4} = 0 \pmod{B} \quad (10)$$

for all $i \geq 2$ and $k \geq 1$. It remains to show that $x_1^d \in B$: we have

$$x_1^d = a_1(x_1^{d-2}) - x_1^{d-2} x_2^2 - x_1^{d-2} x_3^2.$$

The first term lies in B by induction, the others by (9) and (10). This completes the induction and the claim is proved.

Therefore $B = S(M^*)_\chi^H$ as claimed. Further, the generating set must be free, because the Hilbert series of B is bounded above by the Hilbert series of $S(M^*)_\chi^H$ with equality if and only if the generating set is free. $\square$

Corollary 8.

(a) $(S(M^*) \otimes \Lambda^1(M^*))^G$ is generated as an A -module by the set

$$\begin{aligned} \{c_1 &= x_1 y_1 + x_2 y_2 + x_3 y_3, \\ c_2 &= x_2 x_3 y_1 + x_3 x_1 y_2 + x_1 x_2 y_3, \\ c_3 &= x_1^3 y_1 + x_2^3 y_2 + x_3^3 y_3, \\ c_4 &= x_1 x_2^2 y_1 + x_2 x_3^2 y_2 + x_3 x_1^2 y_3, \\ c_5 &= x_2^3 x_3 y_1 + x_3^3 x_1 y_2 + x_1^3 x_2 y_3, \\ c_6 &= x_1^3 x_2^2 y_1 + x_2^3 x_3^2 y_2 + x_3^3 x_1^2 y_3\}. \end{aligned}$$

(b) $(S(M^*) \otimes \Lambda^2(M^*))^G$ is generated as an A -module by the set

$$\begin{aligned} d_1 &= x_1 y_2 y_3 + x_2 y_3 y_1 + x_3 y_1 y_2, \\ d_2 &= x_2 x_3 y_2 y_3 + x_3 x_1 y_3 y_1 + x_1 x_2 y_1 y_2, \\ d_3 &= x_1^3 y_2 y_3 + x_2^3 y_3 y_1 + x_3^3 y_1 y_2, \\ d_4 &= x_1 x_2^2 y_2 y_3 + x_2 x_3^2 y_3 y_1 + x_3 x_1^2 y_1 y_2, \\ d_5 &= x_2^3 x_3 y_2 y_3 + x_3^3 x_1 y_3 y_1 + x_1^3 x_2 y_1 y_2, \\ d_6 &= x_1^3 x_2^2 y_2 y_3 + x_2^3 x_3^2 y_3 y_1 + x_3^3 x_1^2 y_1 y_2. \end{aligned}$$

Theorem 9. R^G is minimally generated as a commutative-graded algebra by

$$\{a_1, a_2, a_3, b, c_1, \dots, c_6, d_1, d_2, d_3, y_1 y_2 y_3\}.$$

Proof. Obviously

$$\{a_1, a_2, a_3, b, c_1, \dots, c_6, d_1, \dots, d_6, y_1 y_2 y_3\}$$

is a generating set. One may check the following relations:

$$\begin{aligned} c_1 c_2 &= -d_4 - a_1 d_1 + d_3, \\ c_1 c_3 &= a_1 d_2 + d_5 - a_2 d_1, \\ c_2 c_3 &= -d_6 + a_1 d_4 - a_1 d_3 + a_2 d_2 - a_3 d_1 - a_1^2 d_1. \end{aligned}$$

So d_4, d_5 , and d_6 are obsolete. To see that the generating set cannot be pruned any further we consider x - and y -degrees separately. Each c_i has y -degree 1 and x -degree $< 6 = \deg(b)$, so it can only possibly be rewritten in terms of a_1, a_2, a_3 , and c_j where $\deg_x(c_j) \leq \deg_x(c_i)$, which is ruled out by freeness over A . Similarly each d_i has y -degree 2 and x -degree $< 6 = \deg(b)$, so it cannot be rewritten in terms of a_1, a_2, a_3, b , and d_j where $\deg_x(d_j) \leq \deg_x(d_i)$ by freeness over A . Further, for d_1 and d_2 , no product of the form $c_i c_j$ has small enough x -degree to rewrite these (except for c_1^2 which is obviously zero). Finally, the first relation above must be the unique expression of $c_1 c_2$ in terms of $A(d_1, d_2, d_3, d_4, d_5, d_6)$ by freeness, so we cannot write d_3 in terms of $c_1 c_2$ and $A(d_1, d_2)$. Since $\deg_x(d_3) = 3$, $c_1 c_2$ is the only product of generators with y -degree one which could be a summand of d_3 . $\square$

References

- [1] Adem, A., Milgram, R. J. (2004). *Cohomology of Finite Groups*. Berlin, Heidelberg: Springer Verlag.
- [2] Anghel, N. (1998). SL_2 -polynomial invariance. *Romanian J. Pure Appl. Math.* 43: 17–46.
- [3] Benson, D. J. (1993). *Polynomial Invariants of Finite Groups*. Cambridge: Cambridge University Press.
- [4] Cartan, H., Eilenberg, S. (1956). *Homological Algebra*. Princeton: Princeton University Press.
- [5] Eddy Campbell, H. E. A., Wehlau, D. L. (2011). *Modular Invariant Theory*. Berlin-Heidelberg: Springer Verlag.
- [6] Derksen, H., Kemper, G. (2008). Computing invariants of algebraic groups in arbitrary characteristic. *Adv. Math.* 217(5):2089–2129.
- [7] Elmer, J. Cohen-Macaulay modules of covariants for cyclic p -groups. Preprint, arXiv:2506.03677v1.
- [8] Hochster, M., Eagon, J. A. (1971). Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci. *Amer. J. Math.* 93:1020–1058.
- [9] Fleischmann, P., Kemper, G., Shank, R. J. (2005). Depth and cohomological connectivity in modular invariant theory. *Trans. Amer. Math. Soc.* 357:3605–3621.
- [10] Meyer, A. (2025). On the modular cohomology of $GL(2, p^n)$ and $SL(2, p^n)$. Preprint, arXiv:2506.04720v1.
- [11] Reiner, V. (1989). Free modules of relative invariants of finite groups. *Stud. Appl. Math.* 81:181–184.
- [12] Swan, R. G. (1960). The p -period of a finite group. *Ill. J. Math.* 4:341–346.