A History Based Logic for Dynamic Preference Updates

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Abstract

History based models suggest a process-based approach to epistemic and temporal reasoning. In this work, we introduce preferences to history based models. Motivated by game theoretical observations, we discuss how preferences can dynamically be updated in history based models. Following, we consider Arrow Update Logic and Event Calculus, and give history based models for these logics. This allows us to relate dynamic logics of history based models to a broader framework.
Keywords History based models, preference logic, dynamic logic, arrow update logic, product update models.

1 Motivation

Formalizing dyadic preferences using Kripke models suggests a one-shot comparison of agents' choices. In this setting, there is no past and the choices are local. When it comes to epistemic and game theoretical situations which require a comparison based on agents' previous actions, such models fall short.

There can be thought of many cases where agents' preferences today depend on their behavior in the past. Simply put, most people prefer chocolate to tofu. Yet, if in the immediate past a person has had a lot of chocolate, his preference for chocolate today may differ. Moreover, if a person develops an intolerance for dairy, then his preference for chocolate can change. A new piece of information may cause agents to update their preferences. Such a situation requires a logic which can describe actions, events, preferences and their updates.

Well-known Kripke structures tend to formalize preferences using state-based discrete models (Hanson, 2001; van Benthem & Liu, 2007; van Benthem, 2014). Most certainly, this approach has some advantages: they are easy to work with and portable to various other modal attitudes. Yet, when it comes to describing preferences that have some dependency on agents' behavior in the past, these models may fall short. For this task, a process based model with the formal strength of expressing temporal and epistemic attitudes is needed (Sack, 2008; Renne *et al.*, 2016). In this paper, take a step towards this direction and offer an alternative formalism to describe subjective preferences based on agents' behavior in the past, or their *histories.* As such, our approach suggests an evolutionary perspective on preferences, immediately allowing the possibility of updating preferences dynamically, which we describe later on.

This is where game theoretical motivations become relevant to our discussion for two main reasons. First, the past behavior or experience of agents affects their future behavior, hence next moves, and consequently becomes an essential part of strategizing. Therefore, having a model with the syntactic strength to express histories can be useful for game theoretical applications. Second, preferences (hence strategies) may depend on agents' histories. What an agent prefers today may be traced back to what he preferred yesterday. Such situations can be described by histories which can express preferences and their revisions. In order to do justice to the subject, in this work we focus only on the formal aspects of the logics in question and leave the game theoretical applications to future work.

Logical structures which rely on histories are not obscure, and luckily, we do not need to look far to find examples. *History based structures*, proposed by Parikh and Ramanujam (Parikh & Ramanujam, 2003), suggest a formal framework that lies between process models and temporal epistemic logics. Epistemic and temporal reasoning in history based models depend on sequences of events, called *histories*. These models have been used to model epistemic messages and communication between agents using a dynamic logical framework, and deontic obligations (Parikh & Ramanujam, 2003; Pacuit, 2007; Pacuit *et al.*, 2006). Furthermore, they are technically similar to interpreted systems which were suggested to formalize temporal and epistemic aspects of program runs (Halpern *et al.*, 2004; Pacuit, 2007). Nevertheless, preferences or any game theoretical formalism have not yet been adequately introduced to history based structures. This is what we achieve in this work.

This work positions itself within the domain of logic of games and process models. The literature abounds focusing on logics for games and preferences on one hand, and logics for processes on the other. (van Benthem, 2014) presented a modern modal logical approach to the subject. A survey of preference logics is given in (Hanson, 2001). Recently, Osherson and Weinstein proposed a logic of preference based on the reasons to form preferences (Osherson & Weinstein, 2012). Within the program of dynamic logic, preference updates have been studied rigorously (van Benthem & Liu, 2007). Nevertheless, these frameworks use Kripke models and suffer from the issues we described earlier.

On the other hand, process models using histories or *runs* were developed to describe logics for programs and their epistemologies (Halpern *et al.*, 2004; Fagin *et al.*, 1995; Fagin *et al.*, 1991; Fagin *et al.*, 1999). Various models for distributed computing have been used in game theoretical formalism and we refer the reader to a brief survey for an overview (Halpern, 2008). To the best of our knowledge, such models have not been extended to express preferences over program runs.

In this work, we extend history based structures by introducing preference modalities — first a static one followed by a dynamic one. By achieving this, we relate the subject to the domains of preference logic and distributed multi-agent systems. After noting the fundamental properties of the preference modality, we proceed to describe various strategies to dynamically update preferences. Following, we focus on some immediate applications of our models and conclude by pointing out the potential use of this framework in various game theoretical situations.

The technical work we present has some overreaching game theoretical implications. In games, strategies are conceived before the play (Hodges, 2013). Any revision or update in strategies is essentially considered as part of the strategies. Similarly, considering the epistemics of games, "the entire stream of beliefs of a player" was summarized in a single entity, which Harsanyi called the player's *type* (Brandenburger, 2014; Harsanyi, 1967). Therefore, any potential (epistemic or preference) update is thought to be contained in the type, which renders dynamic epistemology rather dispensable for game theory. Preferences are also considered in a similar fashion in traditional game theory. Any change or update in agents' subjective preferences are thought to be already included in the initial preferences of the agents. The current paper, however, offers various descriptions to update preferences and consequently strategies. Such a move is helpful to understand the ontology of strategies and whether strategies should be defined statically or dynamically (Başkent, 2011). Therefore, our formalism allows a nuanced examination of various game theoretical and logical issues, and sets a formal basis for the analysis of dynamic preferences in games.

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This paper is organized as follows. First, we introduce the fundamentals of history based models and observe few modally undefinable properties. Following, we introduce a preference modality to history based models and observe how it commutes with other modal attitudes in our language. Next, we motivate why and how preferences can be updated, especially for game theoretical reasons. This allows us to construct a logical structure with preference updates. We show the completeness and decidability of our system. Consequently, we observe how this new logic relate to certain other dynamic logics in the literature. This allows us to further explore the formal qualities of our logic. Finally, we conclude with some remarks on how our work positions itself in the literature.

2 Basic Logical Structure

Different from Kripke models, history based models are constructed by using a given set of events for (preferably, multiple) agents. Events can be seen as actions or moves (in a game theoretical context) which take place over time and potentially affect agents' knowledge. When a history is considered as a sequence of events, it is important to tell apart which events were carried out by which agents, and which agents can see which events and possess what knowledge.

History based structures are constructed by using a set of events E and a set of agents A. For each agent $i \in \mathsf{A}$, $\mathsf{E}_i \subseteq \mathsf{E}$ is the set of events which can be performed or "seen" by agent i. A string h is a *history* over a set of events E, if it is a finite sequence of events in E. For a set of events E, E^* denotes the set of finite strings over E. Similarly, E^ω denotes the set of infinite strings over E.

By lowercase letters $h, h', \ldots$, we denote finite histories from E^*, and by uppercase letters $H, H', \ldots$, we denote those from $\mathsf{E}^* \cup \mathsf{E}^\omega$. A history h is local for agent i, if $h \in \mathsf{E}_i^*$. If a history H is of at least length l, and $m \leq l$, then let $H(m)$ be the m-th element of the sequence. The concatenation of finite history h with (possibly infinite) history H will be denoted by hH. Similarly, for a history of length greater than k or infinite, H_k denotes the finite prefix of H of length k. If a history H prefixes history

H', we write this as $H \leq H'$. A history H' is the *sub-history* of H if each event in H' appears in the same order in H, and it is denoted as $H' \sqsubseteq H$. For example, $ab \leq abc$ but $ac \sqsubseteq abc$.

For any set of histories $\mathcal{H}$, the set $\mathsf{FinPre}(\mathcal{H})$ denotes the set of finite prefixes of the histories in $\mathcal{H}$. More precisely, $\mathsf{FinPre}(\mathcal{H}) = \{h : h \leq H, H \in \mathcal{H}\}$. A set of histories $\mathcal{H}$ is called a *protocol* if it is closed under prefixes, i.e. $\mathsf{FinPre}(\mathcal{H}) \subseteq \mathcal{H}$.

Now we can discuss temporal and epistemic operators in this framework. Given an agent i and a global history H, the agent i can only access some of H. For two histories H, H', if the agent can access the same parts of H and H', then H and H' are indistinguishable for i.

Definition 2.1. Let i be an agent, and $\mathcal{H}$ be a set of histories. A function λ_i : $\mathsf{FinPre}(\mathcal{H}) \mapsto \mathsf{E}_i^*$ is a locality function for i in $\mathcal{H}$.

There are various additional properties one may wish to require of locality functions. An agent i's local clock is *consistent with the global clock* if, for all $H \in \mathcal{H}$ and time points $t, m \in \mathbb{N}$, if $t \leq m$, then $\lambda_i(H_t) \leq \lambda_i(H_m)$ (Pacuit, 2007). An agent i's locality function is *embeddable* if, for all $H \in \mathsf{FinPre}(\mathcal{H})$, $\lambda_i(H) \sqsubseteq H$, that is, all of the events in $\lambda_i(H)$ occur in H, in the same order (Pacuit, 2007). From this point on, we assume all agents' local clocks are consistent with the global clock and $\lambda_i(H) \sqsubseteq H$ for all agents i. In other words "agents are not wrong on about the events that they witness" [ibid]. It is important to notice that the above additional conditions which we imposed on history based models are not obvious in Kripke models. Therefore, varying such conditions would permit us to define different temporal concepts of knowledge and agency. For example, agents' knowledge which is out-of-sync of the global clock allows us to trace timed signals, which can be important for various game theoretical situations.

Definition 2.2. Let i be an agent, and let λ_i be its locality function. Histories h and h' are indistinguishable for agent i, written $h \sim_i h'$, if and only if h and h' are finite histories, and $\lambda_i(h) = \lambda_i(h')$.

For obvious reasons, $\sim_i$ is an equivalence relation. Thus, the epistemic logic of history based structures is the standard multi-agent epistemic logic $\mathbf{S5}_n$.

Given a set of propositional variables P, we define the syntax of history based structures in the Backus - Naur form as follows.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid K_i\varphi \mid \bigcirc\varphi \mid \varphi U \varphi$$

where $p \in P$, $i \in \mathsf{A}$. The knowledge operator for agent i is denoted by K_i and the temporal next-time operator is denoted by $\bigcirc$. We call U the *until operator*. We take implication $\rightarrow$ as an abbreviation in the usual sense. Valuation function V is defined as $V : P \mapsto \wp(\mathsf{FinPre}(\mathcal{H}))$.

A tuple $M = (\mathsf{E}, \mathcal{H}, \mathsf{A}, \mathsf{E}_1, \ldots, \mathsf{E}_n, \lambda_1, \ldots, \lambda_n, V)$ is a history-based temporal-epistemic model, or a *history-based model* for short, where E is a global set of events, $\mathcal{H} \subseteq \mathsf{E}^* \cup \mathsf{E}^\omega$ is a protocol, A is a set of agents, E_i and λ_i are agent i's local event set and locality function respectively, and V is a valuation function as defined earlier.

Truth in history based models is defined inductively as follows. We define semantics for infinite histories (Pacuit, 2007)[1].

$$
\begin{array}{lll}
H, t \models_M p & \text{iff} & H_t \in V(p), \\
H, t \models_M \neg\varphi & \text{iff} & H, t \not\models_M \varphi, \\
H, t \models_M \varphi \wedge \psi & \text{iff} & H, t \models_M \varphi \text{ and } H, t \models_M \psi, \\
H, t \models_M \varphi \vee \psi & \text{iff} & H, t \models_M \varphi \text{ or } H, t \models_M \psi, \\
H, t \models_M \bigcirc\varphi & \text{iff} & H, t+1 \models_M \varphi, \\
H, t \models_M K_i\varphi & \text{iff} & \text{for all } H' \in \mathcal{H}, H_t \sim_i H'_t \text{ implies } H', t \models_M \varphi, \\
H, t \models_M \varphi U \psi & \text{iff} & \text{there exists } k \geq t \text{ such that } H, k \models_M \psi \text{ and,} \\
 & & \text{for all } l, t \leq l < k \text{ implies } H, l \models_M \varphi.
\end{array}
$$

The dual of the epistemic modality is denoted with L_i and defined in the usual way. The expression $M \models \varphi$ denotes the truth of φ in a history based model M, independent from the current history and time-stamp. When it is clear from the context, we will omit the subscript M for the model.

The axioms for history based models are given in the following (Parikh & Ramanujam, 2003; Halpern *et al.*, 2004).

- All tautologies of propositional logic,
- $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$,
- $K_i\varphi \to \varphi \wedge K_iK_i\varphi$,
- $\neg K_i\varphi \to K_i\neg K_i\varphi$,
- $\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$,
- $\bigcirc\neg\varphi \leftrightarrow \neg \bigcirc \varphi$,
- $\varphi U \psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi U \psi))$.

The rules of inference are modus ponens, and normalization for all three modalities:

- $\vdash \varphi, \varphi \to \psi \therefore \vdash \psi$,
- $\vdash \varphi \therefore \vdash K_i\varphi$,
- $\vdash \varphi \therefore \vdash \bigcirc\varphi$,
- $\vdash \varphi \to (\neg\psi \wedge \bigcirc\varphi) \therefore \vdash \varphi \to \neg(\varphi U \psi)$.

[1] This is important in order to prevent some technical problems such as evaluating the truth of a formula at time point n, for example, when the finite history h is shorter than n. We are thankful to the anonymous referee for pointing this out.

For the completeness and complexity results of this system we refer the reader to (Fagin *et al.*, 1995; Halpern *et al.*, 2004; Pacuit, 2007). The standard completeness proof for the systems of epistemic and temporal systems proceed by forming consistent closure sets, very much similar to the procedural completeness proofs of modal logic. Such systems are easy to translate to history based models, as observed by Pacuit, proving their completeness.

History based models combine epistemic and temporal modalities in a complex way. What properties can or cannot be defined using the syntax of history based structures is a meaningful question. In order to answer this question, we need to define bisimulations.

Definition 2.3. For history based models M, M', and history-time pairs H, t in M and H', t' in M', a bisimulation $\bowtie$ between H, t and H', t' is a tuple $\bowtie = (\bowtie_0, \bowtie_1)$ where $\bowtie_0$ is a binary relation between the history-time pairs in M and M' and $\bowtie_1$ is a binary relation between the pairs of history-time pairs in M and M' such that

Propositional base case:

- If $H, t \bowtie_0 H', t'$, then H, t and H', t' satisfy the same propositional variables,

Temporal forth case:

- If $H, t \bowtie_0 H', t'$ and $t < u$, then there is u' in M' such that $t' < u'$, $H, u \bowtie_0 H', u'$ and $(H, t), (H, u) \bowtie_1 (H', t'), (H', u')$,
- If $(H, t), (H, u) \bowtie_1 (H', t'), (H', u')$ and if there is v' with $t' < v' < u'$, then there exists v such that $t < v < u$ and $H, v \bowtie_0 H', v'$,

Temporal back case:

- If $H, t \bowtie_0 H', t'$ and $t' < u'$, then there is u in M such that $t < u$, $H, u \bowtie_0 H', u'$ and $(H, t), (H, u) \bowtie_1 (H', t'), (H', u')$,
- If $(H, t), (H, u) \bowtie_1 (H', t'), (H', u')$ and if there is v with $t < v < u$, then there exists v' such that $t' < v' < u'$ and $H, v \bowtie_0 H', v'$,

Epistemic forth case:

- If $H, t \bowtie_0 H', t'$ and $H_t \sim_i K_l$, then there is K', l' in M' such that $K, l \bowtie_0 K', l'$ and $H'_{t'} \sim_i K'_{l'}$,

Epistemic back case:

- If $H, t \bowtie_0 H', t'$ and $H'_{t'} \sim_i K_{l'}$, then there is K, l in M such that $K, l \bowtie_0 K', l'$ and $H_t \sim_i K_l$,

In the above definition, the *interval bisimulations* defined for the temporal cases are needed for the temporal until modality, as the until modality is essentially an interval process equivalence. Based on this definition, we give the following theorem.

Theorem 2.4. *For history based models M, M', and history-time pairs H, t in M and H', t' in M', if $H, t \bowtie H', t'$, then they satisfy the same formula.*

Proof. For the standard epistemic case see (Blackburn *et al.*, 2001), for the temporal case see (Kurtonina & de Rijke, 1997). ∎

We now observe that some properties of histories are not (modally) definable.

Proposition 2.5. *The length of histories is not modally definable.*

Corollary 2.6. *The finiteness of histories is not modally definable.*

The proof of the above proposition uses bisimulations and the fact that these models are not *backward-looking*. It is possible to construct two bisimilar history-time pairs, one of length n and the other of length $n+k$ for some $k \neq 0$. An easy way to achieve this is to prefix a history of length n with strings of length k. In that case, for instance, we may have bisimilar history-time pairs of $abc, 2 \bowtie xxabc, 4$ with different length - provided that the epistemic relation is defined respecting the bisimulation. Henceforth, if there was a formula defining the length of histories, then this formula must have been preserved under bisimulations. However, this is not possible as the length of the history abc is strictly less than that of $xxabc$.

The above results present a limitation on history based models. For example, the evolutionary aspects of knowledge acquisition cannot be quantitatively described as the length of histories are not modally definable. Such issues can be important in certain game theoretical situations.

3 Preferences in History Based Models

One of the main contributions of this paper is to introduce subjective preferences to history based models with game theoretical applications in mind. This will allow us to describe some basic game theoretical situations using history based models. Furthermore, we will also suggest various ways to update such subjective preferences.

For an agent i, and possibly infinite histories H, H', the expression $H \preceq_i H'$ denotes that "the agent i (weakly) prefers H' to H". The preference relation will be taken as a pre-order satisfying reflexivity and transitivity (not necessarily total) (van Benthem, 2014; Hanson, 2001). If $H \preceq_i H'$ and $H' \preceq_i H$, we denote it by $H \approx_i H'$. The strong preference relation is denoted by $\prec_i$ for agent i, and defined as expected: $H \prec_i H'$ iff $H \preceq_i H'$ and $H \neq H'$.

In order to describe preferences in a modal language, we augment the syntax of the logic of history based structures with a modal operator $\Diamond_i$. In this context $\Diamond_i \varphi$ reads that there is a history which is at least as good as the current one and satisfies φ for agent i. The syntax of history based preferences $\mathcal{L}$ is given as follows.

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid K_i\varphi \mid \bigcirc\varphi \mid \varphi U \varphi \mid \Diamond_i \varphi$$

where $p \in P$, $i \in \mathsf{A}$. We take implication $\rightarrow$ as an abbreviation in the usual sense.

The semantics for the preference modality is given as follows.

$$H, t \models \Diamond_i \varphi \quad \text{iff} \quad \exists H'.\, H \preceq_i H' \text{ and } H', t \models \varphi$$

The dual of the preference modality is denoted by $\Box_i$ and defined in the usual sense. Formally, history based preference model is a tuple

$$M = (\mathsf{E}, \mathcal{H}, \mathsf{A}, \mathsf{E}_1, \ldots, \mathsf{E}_n, \lambda_1, \ldots, \lambda_n, \preceq_1, \ldots, \preceq_n, V)$$

where $\preceq_i$ is the preference comparison order for agent i and the rest is as before.

As we underlined, this formalism compares histories as opposed to propositions. However, it is possible to express preferences over propositions by referring to the histories which satisfy them. The formula $M \models \varphi \rightarrow \Diamond_i \psi$ denotes that the agent i weakly prefers ψ to φ in model M. In other words, each φ has an alternative history which is at least as preferable as the current one and satisfies ψ, thus ψ is weakly preferred to φ.

We take the preference modality as S4 with the expected rule of inference - that is necessitation. For the completeness of our treatment, the axiomatization of history based preference logic is given as follows.

- All tautologies of propositional logic,
- $K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi)$,
- $K_i\varphi \rightarrow \varphi \wedge K_iK_i\varphi$,
- $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$,
- $\Box_i(\varphi \rightarrow \psi) \rightarrow (\Box_i\varphi \rightarrow \Box_i\psi)$,
- $\Box_i\varphi \rightarrow \varphi$,
- $\Box_i\varphi \rightarrow \Box_i\Box_i\varphi$,
- $\bigcirc(\varphi \rightarrow \psi) \rightarrow (\bigcirc\varphi \rightarrow \bigcirc\psi)$,
- $\bigcirc\neg\varphi \leftrightarrow \neg \bigcirc \varphi$,
- $\varphi U \psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi U \psi))$.

The rules of inference are modus ponens, and necessitation for all three modalities:

- $\vdash \varphi, \varphi \rightarrow \psi \therefore \vdash \psi$,
- $\vdash \varphi \therefore \vdash K_i\varphi$,
- $\vdash \varphi \therefore \vdash \Box_i\varphi$
- $\vdash \varphi \therefore \vdash \bigcirc\varphi$,
- $\vdash \varphi \rightarrow (\neg\psi \wedge \bigcirc\varphi) \therefore \vdash \varphi \rightarrow \neg(\varphi U \psi)$.

We call the logic of history based structures with preferences HBPL after *history based preference logic*.

3.1 Expressive Power

In what follows, we introduce additional axioms to express various epistemic and strategic situations in HBPL. They give a direct illustration for HBPL's potential use in game theoretical formalism.

Connectedness of Preferences The connectedness property for the preference relation suggests that any two histories are comparable. The corresponding relational formula is given as $\forall H, H'.H \preceq_i H' \vee H' \preceq_i H$. The corresponding modal axiom is as follows: $\Box_i(\Box_i p \to q) \vee \Box_i(\Box_i q \to p)$. This renders the frame with preference modality as a total pre-order. A total pre-order of preferences over histories equips players with the game theoretical strength to compare each and every history. Philosophically, this allows us to discuss counterfactual reasoning in games, which falls outside the scope of the current paper.

Epistemic Perfect Recall Agents with perfect recall retain knowledge once they acquire it. The standard axiom for this property is given as follows: $K_i \bigcirc \varphi \to \bigcirc K_i \varphi$. It is rather easy to show that this axiom is valid in HBPL. Given an arbitrary history H and a time-stamp t, we start with assuming $H, t \models K_i \bigcirc \varphi$. Our aim is to show that $\bigcirc K_i \varphi$ holds at H, t. Now, by definition, $\forall H'.(H \sim_i H' \to H', t \models \bigcirc \varphi)$. Unfolding the temporal modality gives $\forall H'.(H \sim_i H' \to H', t+1 \models \varphi)$. Now, we can fold back, but this time starting with the epistemic modality. By definition, we first obtain $H, t+1 \models K_i \varphi$, which produces $H, t \models \bigcirc K_i \varphi$. Thus, $K_i \bigcirc \varphi \to \bigcirc K_i \varphi$ is valid in HBPL.[2]

The satisfaction of epistemic perfect recall perhaps justifies the reason why we call sequences of events "histories" (as opposed to *runs* or *traces*). From a philosophical angle, it suggests a strong epistemic restriction on the potential class of games to which history based models might be applicable.

Preferential Perfectness By *preferential perfectness*, we mean that agents do not change their preferences in time. Consider the scheme $\Box_i \bigcirc \varphi \to \bigcirc \Box_i \varphi$. It is also easy to show that this scheme is valid in HBPL. Similar to Perfect Recall, Preferential Perfectness suggests that once acquired preferences do not change. This suggests that, if necessary, they can only be revised at the level of the model. We will use this observation to motivate a dynamic approach to preference change.

Epistemic Rationality By a slight abuse of terminology we call the formula $\Diamond_i K_i \varphi \to K_i \Diamond_i \varphi$ the *Church-Rosser Property*. The HBPL frames which satisfy the Property enjoy the following relational condition:

$$\forall H'H''.(H \preceq_i H' \wedge H \sim_i H'') \to \exists J.(H' \sim_i J \wedge H'' \preceq_i J)$$

In the following we show that the above relational (frame) condition corresponds to the formula $\Diamond_i K_i \varphi \to K_i \Diamond_i \varphi$ in history-based models.

[2]However, the axiom $K_i \bigcirc \varphi \to \bigcirc K_i \varphi$ is not sufficient to establish the completeness of frames with respect to perfect recall (van der Meyden & shu Wong, 2003; van der Meyden, 1994). The additional axiom required for this task is a complicated one:
$K_i \varphi_1 \wedge \bigcirc(K_i \varphi_2 \wedge \neg K_i \varphi_3) \to \neg K_i \neg((K_i \varphi_1) U ((K_i \varphi_2) U \neg \varphi_3))$.

Let the frame condition satisfied in history-based frames. For an arbitrary history-time pair H, t, we assume $H, t \models \Diamond_i K_i \varphi$. We will show that $H, t \models K_i \Diamond_i \varphi$ by *reductio*. First, by the assumption, there exists a history H' such that $H \preceq_i H'$ with $H', t \models K_i \varphi$. Then, by definition, for all J with $H' \sim_i J$, we have $J, t \models \varphi$. Now, in order to obtain a contradiction, assume $H, t \models \neg K_i \Diamond_i \varphi$. By definition, this is equivalent to $H, t \models L_i \Box_i \neg\varphi$. Then, there exists H'' such that $H \sim_i H''$ with $H'', t \models \Box_i \neg\varphi$. Now, we have $H \preceq_i H'$ and $H \sim_i H''$. By the Church-Rosser property, there exists W with $H' \sim_i W$ and $H'' \preceq W$. Now, for all J' with $H'' \preceq_i J$, we have $J', t \models \neg\varphi$. This is a universal statement for all J'' ranging over $H'' \preceq_i J$. Thus, it applies to W. Thus, $W, t \models \neg\varphi$. Similarly, we had for all J with $H' \sim_i J$ that $J, t \models \varphi$. This is a universal statement as well which can be instantiated with W. Thus, $W, t \models \varphi$. This is a contradiction. Thus, $H, t \models K_i \Diamond_i \varphi$, yielding $H, t \models \Diamond_i K_i \varphi \rightarrow K_i \Diamond_i \varphi$.

The converse direction is similar. Assume, $H, t \models \Diamond_i K_i \varphi \rightarrow K_i \Diamond_i \varphi$ for arbitrary H, t and φ. Now, let $H \preceq_i H'$ and $H \sim_i H''$. We will establish by *reductio* that there exists a history J with the desired property. So let us assume that there is no J with $H' \sim_i J$ and $H'' \preceq_i J$. The Church-Rosser formula is a frame-property, thus it is satisfied in every valuation. Let us pick a simple one. For this, we start with assuming that φ is a propositional variable and $H, t \models \Diamond_i K_i \varphi$. Thus, for some $H \preceq_i H'$, we have $H', t \models K_i \varphi$. Thus, for all X with $H' \sim_i X$, we have $X, t \models \varphi$. Based on this observation, we define a minimal valuation V for the propositional variable φ as follows: $V(\varphi) : \{H, t : H' \sim_i H\}$. Now, since we had $H \sim_i H''$ and $H, t \models K_i \Diamond_i \varphi$, we observe that $H'', t \models \Diamond_i \varphi$. Thus, we need a history J' with $H'' \preceq_i J'$ and, by the definition of the minimal valuation for φ, $H' \sim_i J'$. However, by our initial assumption, there is no such J'. Thus, $H'', t \models \Diamond_i \varphi$ cannot be true. Contradiction shows that there is such a history J with $H' \sim_i J$ and $H'' \preceq_i J$.

The Church-Rosser Property suggests that if, at a better history, an agent knows φ, then the agent knows that at a better history φ is satisfied. This has some potential applications in games. For example, elimination of *strictly* dominated strategies is a solution concept in game theory which works irrespective of the order of elimination. Having Church-Rosser property satisfied in epistemic game models suggests that the order of eliminating the strictly dominated strategies does not matter (Leyton-Brown & Shoham, 2008). This is how history models directly relate to game theoretical reasoning.

It is also easy to see the validities of various some other modal formulas, including $\Box_i K_i \bigcirc \varphi \rightarrow \bigcirc \Box_i K_i \varphi$ or $K_i \Box_i \bigcirc \varphi \rightarrow \bigcirc K_i \Box_i \varphi$. Similarly, various commutativity properties of the modalities, such as $K_i K_j \varphi \leftrightarrow K_j K_i \varphi$, can be studied. We leave them to the reader.

For the completeness and decidability of HPBL, we refer the reader to (Halpern *et al.*, 2004; Halpern & Vardi, 1989) where the completeness of epistemic-temporal logics with runs are given by using maximal consistent sets in the usual sense within a quite involved framework. These results carry over directly to HBPL for two reasons. First, for the preference modality, we have not introduced any interaction axiom between the preference modality and any other modality. Second, epistemic modality is known to be S5 and stronger than the preference modality which is taken as S4. Then, HBPL can be seen as a *fusion* of an (epistemically) S5 history based temporal-

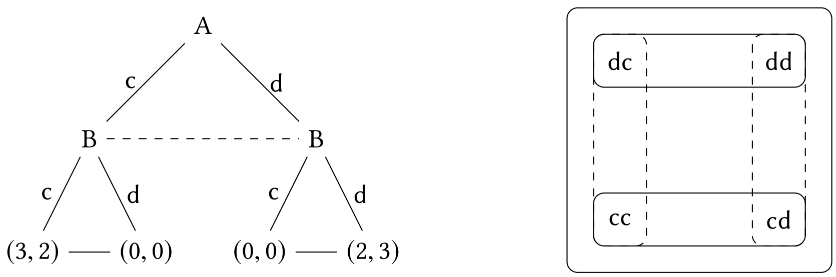

Figure 1: *Battle of the Sexes in extensive normal form (left) and its epistemic indistinguishability relation. The solid line defines the knowledge set of Player A whereas the dashed line defines that of B.*

epistemic logic and an S4 preference logic (Gabbay *et al.*, 2003). Based on these observations and (Halpern *et al.*, 2004; Halpern & Vardi, 1989), the completeness and decidability of HBPL follow.

Theorem 3.1. *HBPL is sound and complete with respect to the axiomatization given.*

Theorem 3.2. *HBPL is decidable.*

4 Dynamic Preference Update on Histories

There can be suggested various reasons as to why players may need to update their preferences in games. They may receive a new piece of information, improve their skills, strategize, or simply make mistakes or cheat. Particularly, in some incomplete information games rational players may be forced to revise their preferences when a new piece of information is introduced. Let us now see a game theoretical example to illustrate our perspective.

We start by considering a variation of *Battle of the Sexes* given in Figure 1. In this coordination game, two players A and B want to attend the same event together. They have two choices: going to a cooking class (c) or dancing (d). Player A prefers the cooking class, whereas Player B prefers dancing. But, both prefer attending the same activity rather than different ones. A game theoretical conundrum occurs, if we are in the situation that A and B made plans to meet up to attend an event together, but they cannot remember where. If they cannot communicate, what should they do?

We assume that each agent has a preference over her preferred activity and they are committed to make the best move based on their preferences. Player A wants to go to the cooking class, B wants to go dancing. Since there is no communication, they do not know each other's move. But, they would cooperate if they learned (in a one-way communication) how the other would act.

We model the game using preferences. As players have symmetric preferences, let us focus only on Player A's (weak) preferences for simplicity. For A, we have

$$\mathsf{cd} \preceq_A \mathsf{dc} \preceq_A \mathsf{dd} \preceq_A \mathsf{cc}$$

and

$$\mathsf{cd} \sim_A \mathsf{cc} \qquad\qquad \mathsf{dd} \sim_A \mathsf{dc}.$$

As we mentioned, A's incentive is to go to cooking class. But, the game is an imperfect information game. Let us assume that A learns that B is on her way to dancing, after a common friend tells her. This eliminates A's preference of going to cooking class. Then, A's highest preference becomes going dancing.

When A learns about B's d move, she revises her preferences to leave only

$$\mathsf{cd} \preceq_A \mathsf{dd}.$$

In this case, her best move becomes d. She no longer prefers going to cooking class over dancing, consequently the preference relation between them is eliminated. Furthermore, for A the epistemic distinguishabilities between cd and cc, and dd and dc are removed, as A now knows the current state of the game. The alternative histories, cc and dc, however, are kept in the model because they may be needed to express various other epistemic situations. In our case, since B may not know that A knows about B's move, we need to keep cc and dc in the model for a complete epistemic description of the situation.

In our model, preference elimination is *controlled* by formulas in the language. For example, after learning about B's move, the preference relation dd $\preceq_A$ cc is eliminated. Because at history dd (at the appropriate time-point), it is true that B makes a d move, whereas at cc (again, at the appropriate time-point), it is *not* the case that B makes a d move. Hence, the fact that "B makes a d move" controls the preference update.

We can generalize from this example. First, we choose to eliminate the preference relations that are made redundant, but keep those histories which were used to define said preferences. One reason is that the update may not be common knowledge and some agents may still consider those histories epistemically plausible. To the best of our knowledge, such preference models have always used Kripkean models to give an account of dynamic preference updates. In this work, we suggest a history-based model for preference updates to take advantage of its applicability to game theoretical situations. Another advantage of using HBPL for game theoretical reasoning is that it admits native tools to express turns and move orders, denoted by the temporal timestamp. In this work, we will not heavily focus on the temporal aspects of the games, but rather discuss the preference dynamics.

Before describing the formal details of dynamic preference updates in history based models, it is important to note that the syntax of HBPL is strong enough to express saddle points and game equilibria in certain cases. In the *Battle of Sexes* example, for instance, the game admits two Nash equilibria: cc and dd (Osborne & Rubinstein, 1994). These histories satisfy certain modal logical properties: they are strictly preferred and there is no strictly better move for the player that his opponent cannot distinguish epistemically. Therefore, we have $\mathsf{cc}, 2 \models \neg\langle \prec_i \cap \sim_{-i}\rangle\top$ and $\mathsf{dd}, 2 \models \neg\langle \prec_i \cap \sim_{-i}\rangle\top$, where $\prec_i$ is the strict preference subrelation of $\preceq_i$, $-i$ is the opponent of i, and $\langle R\rangle$ is the auxiliary modality defined using the relation R. Such a characterization is straightforward for simple games like *Battle of Sexes* or *Prisoner's Dilemma* where information sets are easy to handle and pure strategy equilibria exist. We leave such discussions for future work.

Now, we formally introduce preference updates to history based models.

Following the standard methodology, the preference update will be carried out by a *distinguishing formula* φ. The formula φ is a "distinguishing formula" for H, t and H', t, if $H, t \models \varphi$ but $H', t \models \neg\varphi$. For a given $H \preceq_i H'$, the purpose of a preference update by a distinguishing formula φ is to eliminate $H \preceq_i H'$ from the preference relation so that $H \not\preceq_i H'$ is obtained. We denote the updated preference relation for agent i by $\preceq_i^*$. Syntactically, we denote the preference update by φ with $[\varphi!]$. In the above example, "B makes a d move" acts as the distinguishing formula.

Given a HBPL model $M = (\mathsf{E}, \mathcal{H}, \mathsf{A}, \{\mathsf{E}_i\}_{i\in\mathsf{A}}, \{\lambda_i\}_{i\in\mathsf{A}}, \{\preceq_i\}_{i\in\mathsf{A}}, V)$, the preference update model $M!\varphi$ with respect to the distinguishing formula φ is a tuple

$$M!\varphi = (\mathsf{E}, \mathcal{H}, \mathsf{A}, \{\mathsf{E}_i\}_{i\in\mathsf{A}}, \{\lambda_i\}_{i\in\mathsf{A}}, \{\preceq_i\}_{i\in\mathsf{A}}, \{\preceq_i^*\}_{i\in\mathsf{A}}, V)$$

where the updated preference orders $\preceq_i^*$ are defined as

$$\preceq_i^* := \preceq_i \setminus \{(H, H') : H, t \models_M \varphi \text{ and } H', t \models_M \neg\varphi \text{ for any } t\}.$$

In this formalism, we note that preference updates are independent from the clock, depending only on the distinguishing formulas.

The language $\mathcal{L}^*$ of this system is specified as follows for $p \in P$, $i \in \mathsf{A}$:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid K_i\varphi \mid \bigcirc\varphi \mid \varphi U \varphi \mid \Diamond_i\varphi \mid [!\varphi]\varphi$$

As before, we take implication $\rightarrow$ as an abbreviation.

Given a model M and a distinguishing formula φ, the semantics of the preference update modality is given as follows.

$$H, t \models_M [\varphi!]\psi \quad \text{iff} \quad H, t \models_{M!\varphi} \psi$$

The dual of the $[\cdot!]$ modality is denoted with $\langle\cdot!\rangle$ and defined in the usual sense.

It is now important to observe that $\preceq_i^*$ is indeed a preference order.

Lemma 4.1. *Let $\preceq_i$ be a preference order. For a given φ, the updated relation $\preceq_i^*$ with respect to the distinguishing formula φ is also a preference order.*

Proof. Let $\preceq_i$ be a preference order. Let φ be the distinguishing formula with which we will update the preference relation.

The updated preference $\preceq_i^*$ is reflexive since the model we have is not inconsistent and no history H satisfies a contradictory formula. Therefore, no tuple (H, H) can be removed from $\preceq_i$, which is known to be reflexive. Thus, $\preceq_i^*$ is reflexive as well.

In order to show the transitivity of $\preceq_i^*$, let us consider $H \preceq_i^* H' \preceq_i^* H''$ where $H \not\preceq_i^* H''$. By definition, we have $H \preceq_i H' \preceq_i H''$ and consequently $H \preceq_i H''$. Then, since we assumed that $H \not\preceq_i^* H''$ for the distinguishing formula φ, we have $H, t \models \varphi$ and $H'', t \models \neg\varphi$. Then, for H', t we have two options. It either satisfies φ or $\neg\varphi$. If $H', t \models \varphi$, then φ is the distinguishing formula for the tuples H', t and H'', t, resulting in the elimination of $H' \preceq_i^* H''$. Otherwise, if $H', t \models \neg\varphi$, then, similarly φ acts as the distinguishing formula for the tuples H, t and H', t, resulting in the elimination of $H \preceq_i^* H'$. In either case, we obtain a contradiction. Consequently, $H \preceq_i^* H''$. Thus, $\preceq_i^*$ is transitive.

Hence, $\preceq_i^*$ is a preference order. ∎

The additional set of axioms for the dynamic preference modality is given as follows.

1. $[\varphi!]p \leftrightarrow p$
2. $[\varphi!]\neg\psi \leftrightarrow \neg[\varphi!]\psi$
3. $[\varphi!](\psi \wedge \chi) \leftrightarrow [\varphi!]\psi \wedge [\varphi!]\chi$
4. $[\varphi!](\psi \vee \chi) \leftrightarrow [\varphi!]\psi \vee [\varphi!]\chi$
5. $[\varphi!]K_i\psi \leftrightarrow K_i[\varphi!]\psi$
6. $[\varphi!]\bigcirc\psi \leftrightarrow \bigcirc[\varphi!]\psi$
7. $[\varphi!](\psi U \chi) \leftrightarrow ([\varphi!]\psi)U([\varphi!]\chi)$
8. $[\varphi!]\Diamond_i\psi \leftrightarrow (\neg\varphi \wedge \Diamond_i[\varphi!]\psi) \vee \Diamond_i(\varphi \wedge [\varphi!]\psi)$

The additional proof rule required for the dynamic modality is necessitation, defined in the usual sense: $\vdash \psi \therefore \vdash [\varphi!]\psi$. We call this system HBPL*.

The axioms of HBPL* are sound. The soundness proof is a standard induction on the length of the formulas, hence skipped.

Lemma 4.2. *The axioms of HBPL* are sound.*

Based on this axiomatization, it is easy to see that $[\cdot!]$ is self-dual. In order to see this, consider $[\varphi!]\neg\psi$. By Axiom 2 above, it is equivalent to $\neg[\varphi!]\psi$. Dually, this equals to $\langle\varphi!\rangle\neg\psi$. The choice of φ and ψ was arbitrary, therefore $[\varphi!]\psi \equiv \langle\varphi!\rangle\psi$.

The above axioms system acts as a set of reduction functions from HBPL* to HBPL, reducing the complexity of dynamic preference formulas to the formulas in the language of HBPL.

The Boolean cases for this reduction are immediate. Let us consider the epistemic case in a given model M. We start with $H, t \models_M [\varphi!]K_i\psi$.

$$\begin{array}{lll} H, t \models_M [\varphi!]K_i\psi & \text{iff} & H, t \models_{M!\varphi} K_i\psi \\ & \text{iff} & \forall H'.H_t \sim_i H'_t, H', t \models_{M!\varphi} \psi \\ & \text{iff} & \forall H'.H_t \sim_i H'_t, H', t \models_M [\varphi!]\psi \\ & \text{iff} & H, t \models_M K_i[\varphi!]\psi \end{array}$$

The above argument is sound as the preference updates do not affect the epistemic accessibility relations. The proof of soundness for the reduction axiom for the next-time operator is along the same lines, hence skipped. The until modality is intriguing. We start with $H, t \models_M [\varphi!](\psi U \chi)$.

$$
\begin{array}{lll}
H, t \models_M [\varphi!](\psi U \chi) & \text{iff} & H, t \models_{M!\varphi} \psi U \chi \\
& \text{iff} & \exists k \geq t \,.\, H, k \models_{M!\varphi} \chi \text{ and} \\
& & \forall l \,.\, t \leq l < k,\ H, l \models_{M!\varphi} \psi \\
& \text{iff} & \exists k \geq t \,.\, H, k \models_M [\varphi!]\chi \text{ and} \\
& & \forall l \,.\, t \leq l < k,\ H, l \models_M [\varphi!]\psi \\
& \text{iff} & H, t \models_M ([\varphi!]\psi)U([\varphi!]\chi)
\end{array}
$$

Next, let us consider the axiom for reducing the preference modality. We start with $H, t \models_M [\varphi!]\diamond_i\psi$. After taking φ as a distinguishing formula, we have two options: the current history does not satisfy the distinguishing formula φ *or* the accessible histories do not satisfy $\neg\varphi$. We consider both cases separately.

$$
\begin{array}{lll}
H, t \models_M [\varphi!]\diamond_i\psi & \text{iff} & H, t \models_{M!\varphi} \diamond_i\psi \\
& & (\textit{Case 1}\text{: } \varphi \text{ is not satisfied at the current state}) \\
& \text{iff} & H, t \models_{M!\varphi} \diamond_i\psi \text{ and } H, t \models_M \neg\varphi \\
& \text{iff} & H, t \models_M \neg\varphi \text{ and } \exists H'.H \preceq_i H' \text{ such that} \\
& & H', t \models_{M!\varphi} \psi \\
& \text{iff} & H, t \models_M \neg\varphi \text{ and } H', t \models_M [\varphi!]\psi for H \preceq_i H' \\
& \text{iff} & H, t \models_M \neg\varphi \text{ and } H, t \models_M \diamond_i[\varphi!]\psi \\
& \text{iff} & H, t \models_M \neg\varphi \wedge \diamond_i[\varphi!]\psi \\
& & (\textit{Case 2}\text{: } \neg\varphi \text{ is not satisfied at accessible histories}) \\
& \text{iff} & H', t \models_{M!\varphi} \psi \text{ for } H \preceq_i H' \\
& & (\text{as } H' \text{ cannot satisfy } \neg\varphi \text{ in } M) \\
& \text{iff} & H', t \models_{M!\varphi} \psi \text{ for } H \preceq_i H' \text{ and } H', t \models_M \varphi \\
& \text{iff} & H', t \models_M [\varphi!]\psi \text{ and } H', t \models_M \varphi \text{ for } H \preceq_i H' \\
& \text{iff} & H', t \models_M [\varphi!]\psi \wedge \varphi \text{ for } H \preceq_i H' \\
& \text{iff} & H, t \models_M \diamond_i(\varphi \wedge [\varphi!]\psi) \\
& & (\text{combining } \textit{Cases 1} \text{ and } \textit{2} \text{ disjunctively:}) \\
& \text{iff} & H, t \models_M (\neg\varphi \wedge \diamond_i[\varphi!]\psi) \vee \diamond_i(\varphi \wedge [\varphi!]\psi)
\end{array}
$$

It is important to note that for the case of consecutive updates $[\varphi!][\psi!]\chi$, we do not have a general reduction axiom. Yet, together with the necessitation rule for $[\cdot!]$, we still have a complete axiom system[3], as we will now show.

We call a formula φ *update-free* if it does not contain any subformula with the preference update modality $[\cdot!]$. Our completeness argument rests on a rewriting of any formula to a logically equivalent update-free formula.

[3]For a detailed exposition of such reductions in the context of dynamic epistemic logic, we refer the reader to (Moss, 2015).

Lemma 4.3. *Every formula in HBPL* can be rewritten as a logically equivalent update-free formula.*

Proof. The argument is based on a translation t on the formulas of HBPL* which will mimick the axioms of HBPL* to "simplify" the complex formulas.

By the soundness of the necessitation rule for $[\cdot!]$, we can translate formulas with consecutive updates of type $[\varphi!][\psi!]\chi$ from "inside out" (Moss, 2015). We define the translation function t as follows.

$$
\begin{aligned}
t(p) &= p\\
t(\neg\varphi) &= \neg t(\varphi)\\
t(\varphi \wedge \psi) &= t(\varphi) \wedge t(\psi)\\
t(\varphi \vee \psi) &= t(\varphi) \vee t(\psi)\\
t(K_i\varphi) &= K_i t(\varphi)\\
t(\Diamond_i\varphi) &= \Diamond_i t(\varphi)\\
t(\bigcirc\varphi) &= \bigcirc t(\varphi)\\
t(\varphi U \psi) &= t(\varphi) U t(\psi)\\
t([\varphi!]p) &= p\\
t([\varphi!]\neg\psi) &= \neg t([\varphi!]\psi)\\
t([\varphi!](\psi \wedge \chi)) &= t([\varphi!]\psi) \wedge t([\varphi!]\chi)\\
t([\varphi!](\psi \vee \chi)) &= t([\varphi!]\psi) \vee t([\varphi!]\chi)\\
t([\varphi!]K_i\psi) &= K_i t([\varphi!]\psi)\\
t([\varphi!]\bigcirc\psi) &= \bigcirc t([\varphi!]\psi)\\
t([\varphi!](\psi U \chi)) &= t([\varphi!]\psi) U t([\varphi!]\chi)\\
t([\varphi!]\Diamond_i\psi) &= (\neg\varphi \wedge \Diamond_i t([\varphi!]\psi)) \vee \Diamond_i(\varphi \wedge t([\varphi!]\psi))\\
t([\varphi!][\psi!]\chi) &= t([\varphi!]t([\psi!]\chi))
\end{aligned}
$$

We note that the form of the final case of this definition means that t is not defined by straightforward structural induction. Nevertheless, the function t can be shown to be well-defined by lexicographic induction on the number of updates and the length of the formula. Alternatively, the translation may be formulated by means of two inductively defined functions, mimicking the lexicographic induction; see the appendix for details.

Let us argue here how the translation function works for consecutive updates. Consider $\models \chi$. By soundness of the necessitation rule, $\models [\psi!]\chi$. By the induction hypothesis, $\models t([\psi!]\chi)$ is an update-free formula. By another application of the soundness of the necessitation rule, we obtain $\models [\varphi!]t([\psi!]\chi)$ where $t([\psi!]\chi)$ is update-free. Another application of the t function yields $t([\varphi!][\psi!]\chi) = t([\varphi!]t([\psi!]\chi))$. ■

The completeness of HBPL* follows.

Theorem 4.4. *HBPL* is complete with respect to the axiomatization given.*

Proof. We will prove that for any $\varphi \in \mathcal{L}^*$, it follows that $\vdash \varphi \leftrightarrow t(\varphi)$.

The argument is by lexicographic induction on the number of updates and the complexity of φ. The cases for the Booleans, (static) modals and the update operator are self-evident using proof rules of modal and propositional logic and the induction hypothesis. In what follows, we prove the theorem for $\varphi = [\alpha!][\beta!]\gamma$.

The induction hypothesis says that for all ψ with fewer number of updates than φ, we have $\vdash \psi \leftrightarrow t(\psi)$. Now, let us consider the case $\varphi = [\alpha!][\beta!]\gamma$.

1. $\vdash [\beta!]\gamma \leftrightarrow t([\beta!]\gamma)$
2. $\vdash [\alpha!][\beta!]\gamma \leftrightarrow [\alpha!]t([\beta!]\gamma)$
3. $\vdash [\alpha!]t([\beta!]\gamma) \leftrightarrow t([\alpha!]t([\beta!]\gamma))$
4. $\vdash [\alpha!][\beta!]\gamma \leftrightarrow t([\alpha!]t([\beta!]\gamma))$

A brief narration of the proof is in order. We first start with considering an instance of the induction hypothesis for the formula $[\beta!]\gamma$. Then, by the necessitation with $[\alpha!]$, we obtain Line 2. Now, it is possible to apply the induction hypothesis to $[\alpha!]t([\beta!]\gamma)$ since it has fewer updates than $[\alpha!][\beta!]\gamma$. Next, we obtain Line 3 by the induction hypothesis with the formula $[\alpha!]t([\beta!]\gamma)$. Lines 2 and 3 produce Line 4 by propositional logic. Since $t([\varphi!][\psi!]\chi) = t([\varphi!]t([\psi!]\chi))$ by definition, we conclude the expected result: $\vdash [\alpha!][\beta!]\gamma \leftrightarrow t([\alpha!][\beta!]\gamma)$. ∎

An immediate by-product of the above result is the decidability of HBPL*.

Theorem 4.5. *HBPL* is decidable.*

Proof. The proof follows from Theorems 3.2 and 4.4. ∎

The logic HBPL* presents a logical characterization of dynamic preference updates. A potential application of HBPL* is to analyze game theoretical equilibria in this context. Furthermore, considering the translation between history based models and interpreted systems (Pacuit, 2007), our results can be applied to interpreted systems where program runs with dynamic preferences can be constructed.

5 Further Applications

In what follows, we relate HBPL* to some other well-known dynamic systems and examine how HBPL* may position itself within the domain of dynamic epistemic/temporal logics. We start with Arrow Update Logic and then consider an event calculus to suggest a product update for HBPL*.

5.1 Arrow Update Logic and Dynamic Preference Updates in History Based Models

A recent work on dynamic epistemic logic suggests an alternative formalism to update models (Kooi & Renne, 2011). In this method, called "arrow updates", relational *arrows* are eliminated without eliminating the *states*. Arrow Update Logic (AUL, for short) discusses this methodology in a dynamic epistemic framework and compares it to other dynamic epistemic paradigms. Considering the similarities in the methodologies of both arrow update models and history based models, we carry the discussion over to dynamic preference logic and give a history based model for AUL. This introduces histories into AUL, and advances our study of dynamic preference updates in history based models. As such, we broaden our formal toolkit to discuss game theoretical preferences.

The syntax of AUL is based on the standard language of modal logic augmented by the arrow update modality $[U]$. The arrow update modality depends on a set U, which is constructed as follows, for $i \in \mathsf{A}$:

$$U ::= (\varphi, i, \varphi) \mid (\varphi, i, \varphi) \cup U$$

The set U is called an "arrow specification". In our context it denotes which preferences (that is relational arrows for the preferences) will be kept in the model after an update $[U]$. It is important to notice that the arrow specification sets does not have any semantic closure condition.

Let us now interpret AUL in history based models. Given a HBPL model $M = (\mathsf{E}, \mathcal{H}, \mathsf{A}, \{\mathsf{E}_i\}_{i\in\mathsf{A}}, \{\lambda_i\}_{i\in\mathsf{A}}, \{\preceq_i\}_{i\in\mathsf{A}}, V)$, we define an arrow update model

$$M{\times}U = \{\mathsf{E}^{\times}, \mathcal{H}^{\times}, \mathsf{A}^{\times}, \{\mathsf{E}_i^{\times}\}_{i\in\mathsf{A}}, \{\lambda_i^{\times}\}_{i\in\mathsf{A}}, \{\preceq_i^{\times}\}_{i\in\mathsf{A}}, V^{\times})$$

with respect to a set of arrow specifications U as follows, for $i \in \mathsf{A}$ and for any t.

$$\begin{aligned}
&\mathsf{E}^{\times} := \mathsf{E} && \mathsf{E}_i^{\times} := \mathsf{E}_i\\
&\mathcal{H}^{\times} := \mathcal{H} && \mathsf{A}^{\times} := \mathsf{A}\\
&\lambda_i^{\times} := \lambda_i && V^{\times} := V\\
&\preceq_i^{\times} := \{(H, H') \in \preceq_i \mid \forall t.\exists(\varphi, i, \varphi') \in U : H, t \models_M \varphi \text{ and } H', t \models_M \varphi'\}
\end{aligned}$$

If the arrow specification set U that is used to arrow-update the preference relation $\preceq_i$ needs to be made explicit, we will use the notation $\preceq_i^{\times}[U]$. Furthermore, the quantification on the time stamp t suggests that the arrow update is carried out over all time points. The alternative quantification (that is $\exists(\varphi, i, \varphi').\forall t$) would suggest that once the arrow specification is fixed, it does not change over time. This is a very strong approach to dynamic epistemology and does not address our concerns. In what follows, we will prove our results for an arbitrary time stamp t which can be generalized to all t.[4]

Different from the most dynamic update methodologies, arrow updates specify which relational arrows are to be kept in the model after an update. Yet, this specification is controlled by a set which does not have any semantic closure condition. Consequently, the relation $\preceq_i^{\times}$ obtained after the arrow update is *not* necessarily S4, thus may not be a preference relation. Then, how can we capture preference updates $\preceq_i^{*}$ using arrow updates and specification sets? This is what we focus on in the sequel.

[4]It is important to notice that the time element introduces a second dimension for the epistemic issues we discuss. Different combinations of quantification over histories and time stamps may suggest new approaches to *deletion-based* dynamic logics, including sabotage logics (van Benthem, 2005). As such, the current system can be used to develop extensions of those systems.

Now, let φ be a distinguishing formula which will control the preference update for the preference relation $\preceq_i$ for i. How can we construct an arrow specification set for this update?

We start by considering the following arrow specification sets for the preference relation $\preceq_i$, distinguished formula φ and agent i:

$$\begin{aligned}\text{Arrow}(\varphi, \preceq_i) &:= \{(\neg\varphi, i, \top), (\top, i, \varphi)\}\\ \text{StrongArrow}(\varphi, \preceq_i) &:= \{(\neg\varphi, i, \varphi)\}\end{aligned}$$

Both "Arrow" and "StrongArrow" approximate the updated relation $\preceq_i^*$. We denote the corresponding updated preference relations as "$\preceq_i^\times$[Arrow]" and "$\preceq_i^\times$[StrongArrow]", respectively.

Theorem 5.1. *For a given preference relation $\preceq_i$ in a HBPL* model M, let $\preceq_i^*$ be the updated preference relation with respect to the distinguishing formula φ. Then,*

$$\preceq_i^\times[\text{StrongArrow}] \subseteq \preceq_i^* \subseteq \preceq_i^\times[\text{Arrow}].$$

Proof. Let $(H, H') \in \preceq_i$.

Assume that $(H, H') \in \preceq_i^\times[\text{StrongArrow}]$. Therefore, for any t, we have $H, t \models \neg\varphi$ and $H', t \models \varphi$. Then, by definition, $(H, H') \in \preceq_i^*$. Thus, $\preceq_i^\times[\text{StrongArrow}] \subseteq \preceq_i^*$

Now, assume that $(H, H') \in \preceq_i^*$. Thus, either $H, t \models \neg\varphi$ or $H', t \models \varphi$ for any t. For the prior case, there is a tuple $(\neg\varphi, i, \top) \in \text{Arrow}(\varphi, \preceq_i)$ so that $(H, H') \in \preceq_i^\times$. For the latter case, similarly, there exists a tuple $(\top, i, \varphi) \in \text{Arrow}(\varphi, \preceq_i)$ so that $(H, H') \in \preceq_i^\times$.

Hence, $\preceq_i^* \subseteq \preceq_i^\times[\text{Arrow}]$. ■

Furthermore, we observe that the relation $\preceq_i^\times[\text{Arrow}]$ is S4. This is important because, independent from its relation to $\preceq_i^*$, it suggests a way to construct an S4 arrow update with an explicit arrow specification set. Later on, we will make the connection between $\preceq_i^*$ and $\preceq_i^\times[\text{Arrow}]$ more precise.

Proposition 5.2. *The relation $\preceq_i^\times$[Arrow] is reflexive and transitive.*

Proof. Consider an arbitrary history H at t. If $H, t \models \varphi$, then $(H, H) \in \preceq_i^\times[\text{Arrow}]$ since there is a specification $(\top, i, \varphi) \in \text{Arrow}(\varphi, \preceq_i)$. Otherwise, if $H, t \models \neg\varphi$, then we still have $(H, H) \in \preceq_i^\times[\text{Arrow}]$ as there is a specification $(\neg\varphi, i, \top) \in \text{Arrow}(\varphi, \preceq_i)$. In each case, $\preceq_i^\times[\text{Arrow}]$ is reflexive.

To prove transitivity, take $(H, H') \in \preceq_i^\times[\text{Arrow}]$ and $(H', H'') \in \preceq_i^\times[\text{Arrow}]$. There are four possibilities for the tuples H, H' and H', H'' depending on their arrow specification:

$$\begin{array}{lll}(i) & H, t \models \neg\varphi \text{ and } H', t \models \top & H', t \models \neg\varphi \text{ and } H'', t \models \top\\ (ii) & H, t \models \neg\varphi \text{ and } H', t \models \top & H', t \models \top \text{ and } H'', t \models \varphi\\ (iii) & H, t \models \top \text{ and } H', t \models \varphi & H', t \models \neg\varphi \text{ and } H'', t \models \top\\ (iv) & H, t \models \top \text{ and } H', t \models \varphi & H', t \models \top \text{ and } H'', t \models \varphi\end{array}$$

Now, the case (iii) is impossible as we cannot have $H', t \models \varphi \wedge \neg\varphi$. For the remaining cases, we have $(H, H'') \in \preceq_i^\times[\text{Arrow}]$. For the cases (i) and (ii) we use the specification $(\neg\varphi, i, \top)$ whereas for the case (iv) we use $(\top, i, \varphi)$.

Thus, $\preceq_i^\times[\text{Arrow}]$ is reflexive and transitive. ■

On the other hand, $\preceq_i^{\times}$[StrongArrow] is not reflexive.

Proposition 5.3. *The relation* $\preceq_i^{\times}$[StrongArrow] *is not reflexive.*

Proof. We observe that $(H, H) \notin \preceq_i^{\times}$[StrongArrow] since we cannot have $H, t \models \varphi \wedge \neg\varphi$ for any history-time pair H, t. Thus, $\preceq_i^{\times}$[StrongArrow] is not reflexive, thus cannot produce an S4 modality. ■

Then, the natural question is the connection between and $\preceq_i^*$ and $\preceq_i^{\times}$[Arrow].

Theorem 5.4. *For a given preference relation $\preceq_i$ in a HBPL* model M, let $\preceq_i^*$ be the updated preference relation with respect to the distinguishing formula φ. Let $\preceq_i^{\times}$*[Arrow] *be an arrow-updated preference relation based on the specification set* Arrow$(\varphi, \preceq_i)$ = $\{(\neg\varphi, i, \top), (\top, i, \varphi)\}$ *for the preference relation $\preceq_i^*$.*

Then, $\preceq_i^* = \preceq_i^{\times}$[Arrow].

Proof. We only need to show that $\preceq_i^{\times}$[Arrow] $\subseteq \preceq_i^*$.

Let $(H, H') \in \preceq_i^{\times}$[Arrow] be an arbitrary tuple. Then, there exists an arrow update specification tuple $(\alpha, i, \beta) \in$ Arrow$(\varphi, \preceq_i)$ such that $H, t \models \alpha$ and $H', t \models \beta$ for any t. There are only two possibilities for the arrow specification in Arrow$(\varphi, \preceq_i)$: either $(\alpha, i, \beta) = (\neg\varphi, i, \top)$ or $(\alpha, i, \beta) = (\top, i, \varphi)$.

In the former case, we have $H, t \models \neg\varphi$ and $H', t \models \top$ for any t. Hence, by definition, $(H, H') \in \preceq_i^*$. In the latter case, we have $H, t \models \top$ and $H', t \models \varphi$ for any t. Hence, by definition, $(H, H') \in \preceq_i^*$. In either case, we have the desired result: $\preceq_i^{\times}$[Arrow] $\subseteq \preceq_i^*$.

Thus, $\preceq_i^* = \preceq_i^{\times}$[Arrow]. ■

The above theorem illustrates how it is possible to recover the update methodology of HBPL* using that of AUL by paying special attention to the arrow specifications and restricting them to specific sets.

So far, we have discussed the relational aspects of arrow updates, particularly the way that AUL manipulates the relations. Next, we discuss how the semantics for the arrow updates can be given in history based models. For a specification set U, the arrow update modality $[U]$ can be defined in HBPL* as follows.

$$H, t \models_M [U]\varphi \quad \text{iff} \quad H, t \models_{M \times U} \varphi$$

This definition specifies what remains to be true after an arrow update using a given specification set U. Depending on the syntactic complexity of φ, it is possible to internalize the update using the axioms of AUL.

The axioms of AUL introduce reduction axioms for the dynamic modality $[U]$ and are built on the axioms of classical unimodal logic. In what follows, we only give the axioms governing the arrow update modality where $p \in P$.

- $[U]p \leftrightarrow p$
- $[U]\neg\varphi \leftrightarrow \neg[U]\varphi$
- $[U](\varphi \wedge \psi) \leftrightarrow ([U]\varphi \wedge [U]\psi)$

- $[U](\varphi \vee \psi) \leftrightarrow ([U]\varphi \vee [U]\psi)$
- $[U]K_i\varphi \leftrightarrow \bigwedge_{(\psi,i,\chi)\in U}(\psi \to K_i(\chi \to [U]\varphi))$

It is a straightforward exercise to see the soundness of these axioms in history based arrow update models $M \times U$ in a similar way we have done for HBPL*. We leave it to the reader[5]. Therefore, these axioms are sound for HBPL*.

Now, we observe that from a given HBPL* model, we can obtain the same updated model using both arrow-updates and dynamic preference updates.

Theorem 5.5. *Let* $M = (\mathsf{E}, \mathcal{H}, \mathsf{A}, \{\mathsf{E}_i\}_{i\in\mathsf{A}}, \{\lambda_i\}_{i\in\mathsf{A}}, \{\preceq_i\}_{i\in\mathsf{A}}, V)$ *be a given HBPL* model and* φ *be a formula.*

Then, $M!\varphi = M \times \mathrm{Arrow}(\varphi, \preceq_i)$ *where* $\mathrm{Arrow}(\varphi, \preceq_i) = \{(\neg\varphi, i, \top), (\top, i, \varphi)\}$.

Proof. Follows immediately based on Theorem 5.4. ∎

Arrow updates offer alternative methods to update preferences. For example, consecutive arrow updates can be *composed* and the same methodology can be carried over to preference updates via arrow updates. Let U and U' be two sets of arrow update specifications. The composition of U with U', denoted by $U \circ U'$, is defined as follows (Kooi & Renne, 2011).

Definition 5.6. For two arrow update specifications U and U', the composition $U \circ U'$ is defined as follows:

$$U \circ U' := \{(\varphi \wedge [U]\varphi', i, \psi \wedge [U]\psi') : (\varphi, i, \psi) \in U, (\varphi', i, \psi') \in U'\}$$

Composing arrow updates can be handy. The following axiom of AUL describes how it can be achieved [ibid]:

- $[U][U']\varphi \leftrightarrow [U \circ U']\varphi$

However, composing the individual elements of an arrow update specification set does not necessarily produce the arrow specification set for a preference update.

Example 5.7. Let $U = \{(\neg\varphi, i, \top)\}$ and $U' = \{(\top, i, \varphi,)\}$, separating the specification set $\mathrm{Arrow}(\varphi, \preceq_i)$. Then, $U \circ U' \neq \mathrm{Arrow}(\varphi, \preceq_i)$ and $U' \circ U \neq \mathrm{Arrow}(\varphi, \preceq_i)$. Thus, neither $U \circ U'$ nor $U' \circ U$ is equivalent to $\preceq_i$ for the distinguishing formula φ.

In conclusion, the preference updates of HBPL* cannot be achieved by two consecutive arrow updates specified by U and U'.

Composing certain arrow update specifications allows us to define consecutive updates. Let us see how it works.

Given a preference relation $\preceq_i$, let φ be the distinguishing formula to obtain the updated relation $\preceq_i^*$, and ψ be the distinguishing formula to obtain the updated relation $\preceq_i^{**}$ from $\preceq_i^*$. How can we then characterize the update from $\preceq_i$ to $\preceq_i^{**}$ using both approaches?

[5] We refer the reader to (Moss, 2015) for a general overview of reduction algorithms in dynamic epistemic logic.

In order to apply Theorem 5.1, we start by defining the following arrow specification sets:

$$\text{Arrow}(\varphi, \leq_i) = \{(\neg\varphi, i, \top), (\top, i, \varphi)\}$$
$$\text{Arrow}(\psi, \leq_i^*) = \{(\neg\psi, i, \top), (\top, i, \psi)\}$$

Theorem 5.1 shows that $\text{Arrow}(\varphi, \leq_i)$ and $\text{Arrow}(\psi, \leq_i^*)$ describe the preference updates using the arrow update logic methodology. Therefore, the update $\leq_i \xrightarrow{[\varphi!]} \leq_i^* \xrightarrow{[\psi!]} \leq_i^{**}$ can be characterized by the composition $[\text{Arrow}(\psi, \leq_i^*) \circ \text{Arrow}(\varphi, \leq_i)]$:

$$[\psi!][\varphi!]\chi \leftrightarrow [\text{Arrow}(\psi, \leq_i^*) \circ \text{Arrow}(\varphi, \leq_i)]\chi.$$

The proof of the above claim follows directly from the definitions and Theorem 5.5. We leave it to the reader.

5.2 Product Updates: Histories versus Possible Worlds

An interesting strategy to incorporate Kripke models into history based models is to combine histories/events with states or possible worlds. This approach generates a cartesian product of histories and states, producing a complex and expressive system.

If histories are thought of expressing sequences of events taking place over *time*, then states can be thought of describing them over *space*. Their cartesian product, therefore, describes how histories develop over time and space. This provides histories with extensionality. Such an approach expands the applicability of our logical toolkit to a broader set of games. Furthermore, product updates with histories enables us to express game states independent of history-time pairs. As such, the way that game states and history-time pairs are associated (which is the product-update construction) allows us to reason in and about games from a richer dynamic epistemic framework. Furthermore, from a technical angle, action/product models in dynamic epistemic logic were defined using single events (Baltag *et al.*, 1998). In what follows, we also extend this approach to histories, perceived as sequences of events, suggesting a multi-event based multi-agent system. Therefore, product updates naturally extend event models to history models. From a philosophical angle, this allows us to reason about games beyond single actions and moves and focus on game theoretical behavior, expressed as histories. Product updates of worlds and history-time pairs bring us closer to this goal.

For the completeness of our treatment, let us recall how Kripke models are defined. A Kripke model K is a tuple $K = (W, R, V)$ where W is a non-empty set of states/worlds, R is a binary relation defined on W and $V : P \mapsto \wp(W)$ is a valuation function. In this model, the relation R is a preference relation on the states where wRv means that the state v is preferable to w. It is possible to perceive R as another preference relation or a universal and objective preference ranging over states. We denote the modality associated with R by $\Diamond$. The $\Diamond$ modality will denote the preferences both in the language of the Kripke models and the history models. It will be clear from the context in which system it is used. We denote the unimodal language with the $\Diamond$ by $\mathcal{L}_\Diamond$. Similarly, the updated preference relation R^* is defined as expected in Kripke models: $R^* = R - \{(v, v') : K, v \models \varphi \text{ and } K, v' \models \neg\varphi\}$ for a certain φ which is used to update the model. The update model $K!\varphi$ will have the same domain and valuation. By a slight abuse of notation, the formula $[\varphi!]$ will denote the preference update by the formula φ.

Let us now describe how to incorporate Kripke models into HBPL* to define (preferential) product updates (Gerbrandy, 1999; van Ditmarsch *et al.*, 2007). First, for a given HBPL* model M, we define a *precondition* function pre which assigns a precondition formula in $\mathcal{L}_\diamond$ to history-time tuples (h, t). Since pre is a function, there is a precondition formula for each history-time tuple. From a technical perspective, pre will allow us to take the *product* of history-time pairs and Kripke worlds: the product tuple $(w; h, t)$ of a world w and history-time pair (h, t) will exists if and only if $w \models_K \text{pre}(h, t)$. For clarity in reading, we write $(w; h, t)$ for the product of Kripke state w and the history-time tuple h, t. Now, we define the product update models as follows.

Definition 5.8. Given a HBPL* model M and a Kripke model K, the product update model $M \times K$ is defined as follows:

- The domain is $\{(w; h, t) : w \text{ is a world in } K, h \text{ is a history in } M \text{ and } w \models_K \text{pre}(h, t) \text{ for some } t\}$,
- The new preference relation $R_\leq$ over state/history-time pairs $(w; h, t)R_\leq(w'; h', t)$ is satisfied if and only if wRw' and $h \leq h'$ for some fixed time t;
- The new epistemic relation $R_\sim$ over state/history-time pairs$(w; h, t)R_\sim(w'; h', t)$ is satisfied if and only if wRw' and $h{\sim}h'$ for some fixed time t;
- The new temporal relation R_+ over state/history-time pairs$(w; h, t)R_+(w'; h, t')$ is satisfied if and only if wRw' and $t' = t + 1$ for some fixed history h;
- The product-state $(w; h, t)$ satisfies propositional atom $p \in P$ if and only $w \models_K p$.

In order to keep the technical details at a minimum, the definition above was given for a single agent, and it can be extended to a multi-agent setting using standard methods. Furthermore, the additional relations $R_\leq$, $R_\sim$ and R_+ defined for the product model can also be associated to box-like modal operators $[R_\leq]$, $[R_\sim]$ and $[R_+]$, respectively. In order to keep technical details at a minimum, we skip such details.

The product update in our system is controlled by a new dynamic modality $\langle M; h, t\rangle$. The semantic of the product update modality is given as follows.

$$w \models_K \langle M; h, t\rangle\varphi \quad \text{iff} \quad w \models_K \text{pre}(h, t) \quad \text{implies} \quad (w; h, t) \models_{M\times K} \varphi$$

Notice that for those history-time (h', t') pairs where $(w; h', t')$ is not defined (that is, where $w \not\models_K \text{pre}(h', t')$), the formula $\langle M; h', t'\rangle\varphi$ always holds vacuously at w for any φ.

The completeness of product update of history based models can be proved immediately using the standard methods presented in (Baltag *et al.*, 1998). The standard approach reduces the syntax of product models to that of HBPL. For the completeness of our treatment, let us give the term reduction $\rightsquigarrow$ inductively as follows.

- $\langle M; h, t\rangle p \rightsquigarrow \mathrm{pre}(h, t) \to p$
- $\langle M; h, t\rangle \neg\varphi \rightsquigarrow \mathrm{pre}(h, t) \to \neg\langle M; h, t\rangle\varphi$
- $\langle M; h, t\rangle(\varphi \wedge \psi) \rightsquigarrow \mathrm{pre}(h, t)\varphi \wedge \mathrm{pre}(h, t)\psi$
- $\langle M; h, t\rangle(\varphi \vee \psi) \rightsquigarrow \mathrm{pre}(h, t)\varphi \vee \mathrm{pre}(h, t)\psi$
- $\langle M; h, t\rangle[R_{\leq}]\varphi \rightsquigarrow \mathrm{pre}(h, t) \to \bigwedge\{[R_{\sim}]\langle M; k, t\rangle\varphi : h \leq k\}$
- $\langle M; h, t\rangle[R_{\sim}]\varphi \rightsquigarrow \mathrm{pre}(h, t) \to \bigwedge\{[R_{\sim}]\langle M; k, t\rangle\varphi : h \sim k\}$
- $\langle M; h, t\rangle[R_{+}]\varphi \rightsquigarrow \mathrm{pre}(h, t) \to \bigwedge\{[R_{\sim}]\langle M; h, t'\rangle\varphi : t' = t + 1\}$

Briefly, let us see how the term reduction $\rightsquigarrow$ works for the propositional case. Consider $w \models_K \langle M; h, t\rangle p$ at a state w for a propositional variable p. Then, by definition we have that $w \models_K \mathrm{pre}(h, t)$ implies $(w; h, t) \models_{M\times K} p$. By the construction of the product model, the product state $(w; h, t)$ satisfies a formula if it is constructed by the method we gave in Definition 5.8. Thus, if $w \models \mathrm{pre}(h, t)$, then, again by Definition 5.8, $w \models p$. Therefore, the reduction function $\rightsquigarrow$ translates $\langle M; h, t\rangle p$ to $\mathrm{pre}(h, t) \to p$. The other cases can be shown similarly.

Next, we observe whether various properties of history based models are preserved in the product models.

Lemma 5.9. *History based product models satisfy epistemic perfect recall and preferential perfectness.*

Proof. We present a proof only for the case for epistemic perfect recall, as the proofs for the other cases are similar.

Epistemic perfect recall is defined by the formula $[R_{\sim}][R_{+}]\varphi \to [R_{+}][R_{\sim}]\varphi$. As we have demonstrated earlier, this property is valid in history based models. In order to show that it is also valid in history based product models, we start by assuming $(w; h, t) \models [R_{\sim}][R_{+}]\varphi$ at an arbitrary product state $(w; h, t)$ and a formula φ. Then, by definition, for all product tuples $(w'; h', t)$ with wRw' and $h{\sim_i}h'$, we have $(w'; h', t) \models [R_{+}]\varphi$. Consequently, for all product tuples $(w''; h', t + 1)$ with $w'Rw''$, we have $(w''; h', t + 1) \models \varphi$. Now, by definition, since we have $w'Rw''$ and $h{\sim_i}h'$, we conclude that $(w'; h, t + 1) \models [R_{\sim}]\varphi$. Similarly, since wRw', we can take another step back to deduce $(w; h, t) \models [R_{+}][R_{\sim}]\varphi$. Thus, $(w; h, t) \models [R_{\sim}][R_{+}]\varphi \to [R_{+}][R_{\sim}]\varphi$ for an arbitrary product state $(w; h, t)$.

The proof for preferential perfectness (which is defined by the formula $[R_{\leq}][R_{+}]\varphi \to [R_{+}][R_{\leq}]\varphi$) is similar, thus skipped. ■

Product updates are sometimes labeled as the "most powerful epistemic update calculus" (van Benthem & Liu, 2007). Now, we will take advantage of this powerful tool to illustrate various game theoretical situations in our model.

One of our motivations for introducing preferential product updates over histories is to express various game theoretical situations and notions such as "delayed gratification" (that is the "marshmallow test"), "temptation" and "self-control". The phenomenon of delayed gratification occurs as people tend to prefer good things

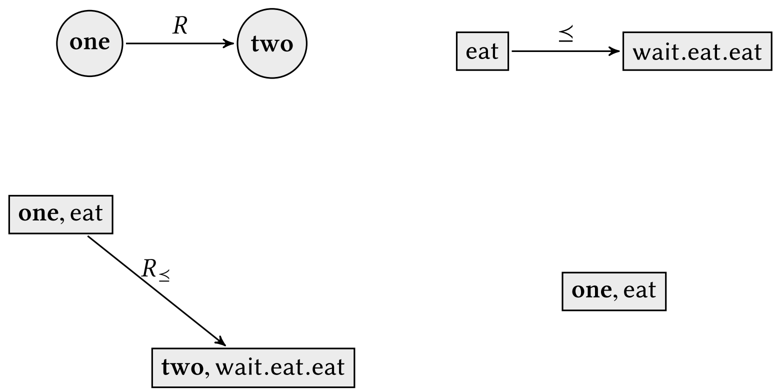

Figure 2: *The Kripke* (upper-left), *the History Based* (upper-right) *Models*; *their Product* (lower-left), *and finally the Updated Product Models.*

now to better things later. The *marshmallow test* is a well-known example for this phenomenon.[6] In this test, children are found to fail at delaying gratification - they choose to have one marshmallow now, rather than waiting for a little while to get an additional marshmallow. Not waiting for a better outcome suggests that the agents' preferences change after a certain time - they rush now and update their preferences and violate their rational commitments. Product updates are ideal candidates to model this situation as they will include states to express the worlds with one and two marshmallows, histories to denote the events of waiting and eating, and a dynamic preference update operator to characterize the preference change, which triggers the agent to have one marshmallow now rather than waiting for the second.

Let us set up the model. The state **one** is the state where there is one marshmallow on the plate and **two** is the state where there are two. Naturally, we have **one**R**two**. We also assume that the players rationally prefer to have a marshmallow later. That is $\mathsf{eat} \preceq \mathsf{wait.eat.eat}$, denoting that the agent prefers to have two marshmallow by waiting. The preconditions pre for the histories are given as follows: $\mathrm{pre}(\mathsf{eat}, 0) = irr$ and $\mathrm{pre}(\mathsf{wait.eat.eat}, 1) = rat$ where irr and rat are propositional formulas, indicating that it is irrational to eat the marshmallow now without waiting and it is rational to wait and then eat more, respectively. Similarly, we have $\mathbf{one} \models irr$ and $\mathbf{two} \models rat$. What happens is, contrary to their game theoretical and rational commitments, the agents update their preferences to have the irrational choice.

However, these two situations are described in different models: one in a Kripke model and the other one in a history based model. The natural question is how to combine them into a single model to express the two-dimensions of preferences: one over the quantities of marshmallow and the other over the histories. This is what our product update achieves.

Let us now see how the failure of the delayed gratification phenomenon can be described by a product update. We have two states one and two, and two histories

[6] See a relevant video at `youtu.be/QX_oy9614HQ`, and the Wikipedia entry at (`en.wikipedia.org/wiki/Stanford_marshmallow_experiment`).

wait and wait.eat.eat, as we mentioned earlier. The proposition *irr* is satisfied at both the world **one** and the history-time pair eat, 0 at time 0; and similarly, *rat* is satisfied at both **two** and wait.eat.eat, 1. Similarly, we have **one**R**two** and eat $\preceq$ wait.eat.eat, yielding the product preference $R_{\preceq}$ as follows, with a slight abuse of notation (See Figure 2):

$$(\mathbf{one}; \text{eat}, 0)\ R_{\preceq}\ (\mathbf{two}; \text{wait.eat.eat}, 1)$$

The product update combines the worlds with histories, respecting the propositional valuations. Yet, as such, it does not express the preference updates. For this, we will use the $[\cdot!]$ operator of HBPL* in order to expresses the breaking point of the agent's preference, yielding a rushed preference over one marshmallow with a shorter history.

The first task is to determine the distinguishing formula which triggers the preference update over world-history pairs. The distinguishing formula is *irr* which symbolizes that the agent suddenly prioritizes being irrational and rushes to having one marshmallow. Notice that *irr* is only satisfied at **one** and eat, 0.

The phenomenon of the marshmallow test, thus, is expressed as follows

$$\mathsf{one}, \mathsf{eat} \models \langle M; \mathsf{eat}, 0\rangle[irr!]\mathsf{noDGC}$$

where the formula noDGC represents the choice induced by the failure of delayed gratification phenomenon – eating the marshmallow now.

Our framework allows various generalizations of the marshmallow test, enriching the discussions regarding the game theoretical interpretation of the phenomenon. One way is to consider a version of the marshmallow test which takes place in the future. In this version, agents are given two choices: one marshmallow at time t in the future or two marshmallows at time $t + k$. In this case, it can be thought that people may favor waiting rather than rushing to having one marshmallow. This hypothetical case contrasts itself very clearly with the original test where the comparison is between now and some time in the future. Another generalization is the case of multiple states with n marshmallows incremented by one at each step. The test of n-marshmallows can be used to identify the cutting-point of where agents give up waiting for more. Our framework can easily express such cases. Such generalizations allow us to consider the interaction between the next-time modality $\bigcirc$ and the dynamic modalities $[\cdot!]$ and $\langle M; h, t\rangle$ in order to have a more nuanced description of the phenomenon of delayed gratification together with its potential extensions and generalizations.

In conclusion, product updates for history based models achieve the following. First, from a technical angle, they extend event-based product models to history-based models, allowing us to experiment with sequences of events and their temporal and epistemic qualities. Second, they incorporate possible worlds into history based models, suggesting a rich area of applications. As we exemplified, the additional formalism becomes necessary to characterize various interesting game theoretical situations.

5.3 Further Approaches to Dynamic Model Updates

The preference update we have introduced forms the basis of further work reported in (Anderson *et al.*, 2016), which introduces a logic for reasoning about uncertain preferences. In that work, history based preference models are extended with *sets of preference relations* for each agent. A corresponding preference update is defined using a set of distinguishing formulae: this update accounts for a change that alters preferences in an uncertain manner. The leading example is (lack of) compliance with newly introduced policies. Agents may have a limited appetite for complying with policies which run counter to their original preferences. The introduction of such a policy will amend the agents' preferences, but the extent to which they will update their decision making is uncertain, so a set of possible new preferences is generated.

(Anderson *et al.*, 2016)'s work is still at a preliminary stage of development: the syntax and semantics of the logic are presented but no proof system or metatheory are developed. Nevertheless this work provides a starting point for an analysis of decision making under uncertainty, or incomplete information reasoning, using history based preference models.

6 Comparison with Other Work

The original work that initiated history based models aims at analyzing the epistemic role of messages (Parikh & Ramanujam, 2003). The authors then briefly discuss Gricean implicatures, motivated by various Wittgensteinian issues. The messages themselves and the way they were defined using histories strongly resembles signaling games. It would not be wrong to claim that the authors have come close to game theoretical concepts, but have not taken the next step. The current paper therefore can be seen as an investigation towards the direction that was implicitly addressed in (Parikh & Ramanujam, 2003).

As we argued, there is an explicit connection between program runs and histories (Pacuit, 2007). Pacuit addressed this relation and showed how histories and runs (in interpreted systems) can be converted to each other. Even if Pacuit himself pointed out a potential game theoretical application of history based models, a clear idea or an application of games with histories was lacking. This motivated our work for introducing preferences to express subjective and game theoretical agency. Clearly, there is further work to be done. A quantitative and utility-based analysis using histories remains to be developed.

A brief discussion of dynamic epistemic and temporal logic goes beyond the scope of this paper. These two frameworks were unified in various ways, almost uniformly using Kripke models (van Benthem *et al.*, 2007). Epistemic-temporal logics are well-studied. It remains, however, to investigate how different systems may help us understand the dynamic interaction between the two as well as how epistemic-temporal modalities become useful in fundamental game theoretical analysis. The current paper attempts at contributing to filling in this gap in the literature.

The literature on interpreted systems is rich (Halpern *et al.*, 2004; Fagin *et al.*, 1995; Fagin *et al.*, 1991; Fagin *et al.*, 1999). It has clear applications in computing and program analysis, which are the underlying motivation for such models. A run of a program can easily be seen as a history with a clear temporal element. From this perspective, introducing *subjective* preferences into interpreted systems may pave the way towards a broader class of applications in system analysis, human-computer interaction and artificial intelligence where programs (and their runs) need to interact with humans with subjective preferences. Such game-like situations require frameworks with preferences, knowledge, temporality and some other similar modalities. Our logic suggests an ideal framework for this direction. Moreover, from a philosophical perspective, it suggests the possibility of introducing preferences into computation. This has the potential to lead to a broader conceptualization of programs, runs and computation.

Modal logical treatment of preferences constitutes a broad literature (Hanson, 2001). A dynamic treatment of preferences was initiated relatively earlier using Kripkean structures (van Benthem & Liu, 2007; van Benthem *et al.*, 2009). Nevertheless, the structural differences of the update models over different models and semantics have not been adequately addressed (van Benthem, 2014). The current paper can therefore be seen as a contribution for this issue. Furthermore, the state based approach to preferences provides a single-shot picture of the situation. As such it fails to portray an *evolutionary* description of preferences. Questions such as "which actions yield what preference?" and "how do preferences evolve over time?" are left unanswered. History based models, however, can identify the sequences of actions (that is "histories") that bring about certain preferences. From a game theoretical perspective, this is an important distinction.

Action models, on the other hand, suggest a very powerful way to combine individual actions with Kripke models and they can easily be extended to produce *product* models with multiple actions (Baltag *et al.*, 1998). Our extension of action models replaces individual actions in the syntactic structure of the model with a sequence of histories, providing a richer and fuller picture. This suggests an immediate application both for action and history models, as we have demonstrated. What remains to be done is to use action based models for epistemic game theory, where moves are considered as actions and the epistemic/temporal aspects are used to define game theoretical equilibria and rationality. Our work can be viewed as a first step towards this direction, where game equilibria can be discussed within our framework.

A recent work on action models extend action models with temporal and epistemic operators (Renne *et al.*, 2016). Their model, however, is based on Kripke models and uses single-shot action models. It also lacks a clear game theoretical motivation and applicability. The current work aims at motivating further work on other epistemic and temporal systems and their applicability to game theory.

Another interesting direction is to study the epistemic and temporal aspects of blockchain protocols using interpreted systems or history based models (Halpern & Pass, 2017). Blockchain protocols are gaining increasing attention and present an intriguing direction for epistemic and temporal logics. It is a matter of interesting debate how preferences would fit in that framework and how agents may have different incentives to deviate from the public ledgers of the blockchain protocol. Our work seems to provide the necessary framework for such inquiries.

The logic HBPL* lacks any methodology to deal with strategies. This is a valid criticism and can be viewed as a design choice. The literature abounds for logics for strategies (Lorini & Moisan, 2011; Ramanujam & Simon, 2008; Başkent, 2011). Our contribution complements this body of work by focusing on preferences rather than strategies. An immediate future work would be to combine them into a unified structure which is expressive enough to describe preferences, game theoretical rationality and strategic reasoning.

7 Conclusion

History based structures have long remained understudied. This paper is an attempt to change this.

We have presented a history based model to reason about changing preferences, with a clear motivation from game theory. Consequently, we have studied how our framework relates to various other well-studied systems in dynamic epistemic logic. This highlights the potential of history based models both in epistemic and game theoretical situations.

What our system lacks is an explicit and detailed study of "time". In our approach, we used time as a mere tracking device or an indicator. It is however possible to use it the way it is instrumentalized in linear temporal logics, with a broader modal toolkit to express more complex situations including indefinite past and future. This creates possible new directions for future work.

Another line of research to extend HBPL* is to consider its modal extensions with deontic and common-knowledge modalities. Such an extension would allow expressing interesting situations where the agents' obligations depend on their past and knowledge, allowing another approach to the "can-ought" problem.

In this paper, we did not take advantage of our system to formalize other game theoretical concepts including strategies and Nash equilibrium. They fall outside the scope of the current paper, yet present future work possibilities.

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Appendix: An Alternative Method to Define the Reduction Function t

In what follows, we present an alternative method the define the reduction function t discussed in the proof of Lemma 4.3.

We define a translation function t as follows

$$\begin{aligned}
t(p) &= p \\
t(\neg\varphi) &= \neg t(\varphi) \\
t(\varphi \wedge \psi) &= t(\varphi) \wedge t(\psi) \\
t(\varphi \vee \psi) &= t(\varphi) \vee t(\psi) \\
t(K_i\varphi) &= K_i t(\varphi) \\
t(\Diamond_i\varphi) &= \Diamond_i t(\varphi) \\
t(\bigcirc\varphi) &= \bigcirc t(\varphi) \\
t(\varphi U \psi) &= t(\varphi) U t(\psi)
\end{aligned}$$

For update free formulas φ, ψ, we define a function T as

$$t([\varphi!]\psi) := T([t(\varphi)!]t(\psi))$$

where T is defined on $[\varphi!]\psi$ for update-free formulas in the following

$$\begin{aligned}
T([\varphi!]p) &= p \\
T([\varphi!]\neg\psi) &= \neg T([\varphi!]\psi) \\
T([\varphi!](\psi \wedge \chi)) &= T([\varphi!]\psi) \wedge T([\varphi!]\chi) \\
T([\varphi!](\psi \vee \chi)) &= T([\varphi!]\psi) \vee T([\varphi!]\chi) \\
T([\varphi!]K_i\psi) &= K_i T([\varphi!]\psi) \\
T([\varphi!]\bigcirc\psi) &= \bigcirc T([\varphi!]\psi) \\
T([\varphi!](\psi U \chi)) &= (T([\varphi!]\psi)) U (T([\varphi!]\chi)) \\
T([\varphi!]\Diamond_i\psi) &= (\neg\varphi \wedge \Diamond_i T([\varphi!]\psi)) \vee (\Diamond_i(\varphi \wedge T([\varphi!]\psi))).
\end{aligned}$$

It is a straight-forward procedural argument to see that t indeed reduces formulas in the language of HBPL^* to update-free formulas.

Let us prove it for the formula $[\varphi!][\psi!]\chi$. By definition we have $t([\varphi!][\psi!]\chi) = T([t(\varphi)!]t([\psi!]\chi))$. By induction hypothesis, $t(\varphi)$ and $t([\psi!]\chi)$ are update-free. Then, T becomes applicable. By induction for T with the update-free formula $t([\psi!]\chi)$, it follows that $T([t(\varphi)!]t([\psi!]\chi))$ is update-free.