

A Kripke Semantics for Monadic BL Chains

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Abstract. We provide a generalisation of Kripke semantics for Monadic first-order Basic Logic of Chains (**MBLC**) of Petr Hájek and prove its soundness and completeness with respect to our semantics. This paper extends the insights of [10] from **BL** to the case of **MBLC**.

Keywords: Substructural logic · Kripke Semantics

1 Introduction

Hájek’s basic logic (**BL**) occupies a central place in contemporary research on fuzzy and substructural logic. First-order **BL** (**BL_{FO}** hereon), like the propositional fragment, is primarily studied algebraically. **BL** is after all the logic of t-norms [6]. The papers of Jipsen and Montagna [8] and Bova and Montagna [1] suggest an alternative view of the situation: We can employ algebraic embedding results via *poset products* to construct generalisations of Kripke semantics appropriate to extensions of **GBL** [10] (and later [3]). This situates systems such as **GBL** and **BL** among constructive and intermediate logics, whose relational semantics are well-understood.

However, we need not restrict our attention to poset product representations of **GBL**-algebras. In fact, the vast majority of representation results in the literature on **BL** and **BL_{FO}** involve *ordinal sums* (and these served as the original inspiration for poset products; see [8]). The question arises what relational semantics might be obtained by ordinal sums as our starting point. Thus we extend our present programme of extracting relational semantics for substructural logics from poset products to ordinal sums, with the monadic fragment of **BL_{FO}** (hereon **MBLC**) as our present case study.

We emphasize that there are numerous advantages in identifying and studying relational semantics in fuzzy logics (and other systems) such as our proposal for **MBLC**. For one, relational semantics can make for interesting examples and counterexamples for logic as alternatives to e.g. counterexamples presented in matrix form. Similarly, having a relational semantics available makes it plausible to pursue semantically-based calculi, such as tableaux or labeled calculi, which exploit insights in the semantics to guide the decomposition of formulas in a

principled way. Finally, relational semantics offer a different intuition on how the logic itself behaves: algebraic models (such as Hájek and his successors) for **BL** and **MBLC** are inherently static. The relational paradigm lends us a different view of logics as inherently dynamic objects (formulas progress in some value according to the behaviour of the poset); and perhaps it is only appropriate that many-valued and fuzzy logics have a means to represent many-values and change of valuation within a fixed structure.

The semantics we devise for **MBLC** is a departure from that of [10] and [9]. The present structures are defined over linear frames, with sloping functions similar to [9], but the worlds (and formulas) of the structure are valued in Wajsberg-hoops. Additionally, the interpretation of quantifiers require a monadic Heyting chain. This presents quite a bit more structure than is the case with [9]. Hence our designation Castaño-Cimadamore-Varela-Rueda, or **CCVR**-structure, as the present structures are derived from representation results via ordinal sums of Wajsberg hoops [2] as opposed to the poset product-based representations of Jipsen et al.

The structure of the paper is otherwise as with [10] or [9]. We give **MBLC**'s Hilbert system, followed by suitable definitions of algebras, validity, and our relational semantics. We prove **MBLC** sound and complete for our semantics, via an isomorphism between the ordinal sum construction and relational structures defined herein.

2 Proof theory for MBLC

We consider briefly the proof theory of **MBLC**. We present the Hilbert-style system for the sake of clarity, but also to serve our later exposition and results (in particular our completeness proof).

The language has a countable number of unary predicate symbols and one individual variable, x , so that atomic formulas are of the form $P_i x$ where $i \in \mathbb{N}$ and $\mathbb{N}$ is the set of natural numbers. The formulas of **MBLC** are inductively defined from atomic formulas and $\perp$, using the binary connectives $\wedge, \vee, \otimes, \rightarrow$, and quantifiers $\forall x$ and $\exists x$ with x the fixed variable. We use $\phi \leftrightarrow \psi$ and $\top$ to abbreviate $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$ and $\perp \rightarrow \perp$, respectively. We will refer to this language as $\mathbb{L}_{\otimes MFO}$, since it extends the language $\mathbb{L}_{MFO}$ of monadic first-order Gödel-Dummett logic (see Note 1) with a second form of conjunction, $\psi \otimes \chi$.

Below, we present the Hilbert system **MBLC_H** of [7].

- (BL1) $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi)).$
- (BL2) $(\phi \otimes \psi) \rightarrow \phi.$ ³
- (BL3) $(\phi \otimes \psi) \rightarrow (\psi \otimes \phi).$

³ Here we do not put $(\phi \otimes \psi) \rightarrow \psi$ as an axiom because of the presence of (BL3) and (BL1).

- (BL4) $(\phi \wedge \psi) \rightarrow (\psi \wedge \phi)$.
- (BL5a) $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \otimes \psi) \rightarrow \chi)$.
- (BL5b) $((\phi \otimes \psi) \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi))$.
- (BL6) $((\phi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \phi) \rightarrow \chi) \rightarrow \chi)$.
- (BL7) $\perp \rightarrow \phi$.
- ($\wedge$ 1) $(\phi \otimes (\phi \rightarrow \psi)) \rightarrow (\phi \wedge \psi)$.
- ($\wedge$ 2) $(\phi \wedge \psi) \rightarrow (\phi \otimes (\phi \rightarrow \psi))$.
- ($\vee$ 1) $(\phi \vee \psi) \rightarrow (((\phi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \phi) \rightarrow \phi))$.
- ($\vee$ 2) $((\phi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \phi) \rightarrow \phi) \rightarrow (\phi \vee \psi)$.
- (M1) $\forall \phi \rightarrow \phi$.
- (M2) $\forall(\phi \rightarrow \forall \psi) \leftrightarrow (\exists \phi \rightarrow \forall \psi)$.
- (M3) $\forall(\forall \phi \rightarrow \psi) \leftrightarrow (\forall \phi \rightarrow \forall \psi)$.
- (M4) $\forall(\exists \phi \vee \psi) \leftrightarrow (\exists \phi \vee \forall \psi)$.
- (M5) $\exists(\phi \otimes \phi) \leftrightarrow (\exists \phi \otimes \exists \phi)$.
- (Chain) $\forall x(\phi \vee \psi) \rightarrow (\forall x \phi \vee \forall x \psi)$.
- (R1) From $\vdash_{\mathbf{MBLC}_H} \phi, \vdash_{\mathbf{MBLC}_H} \phi \rightarrow \psi$ infer $\vdash_{\mathbf{MBLC}_H} \psi$.
- (R2) From $\vdash_{\mathbf{MBLC}_H} \phi$ infer $\vdash_{\mathbf{MBLC}_H} \forall x \phi$.

The system we obtained are referred to as $\mathbf{MBLC}_H$. When we wish to stress the precise system in which a formula ϕ is derivable we use the system as a subscript of the provability sign, e.g. $\vdash_{\mathbf{MBLC}} \phi$.

Note 1. Monadic Gödel-Dummett logic ($\mathbf{MGD}$) results from $\mathbf{MBLC}$ by adding the axiom $(\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi)$.

3 Algebraic Semantics

We situate the algebraic semantics characterising $\mathbf{BL}$ in terms of the somewhat larger theory of residuated lattices.

Definition 1. $\mathbb{A} = \langle A, \wedge, \vee, \otimes, 1, \rightarrow \rangle$ is called a commutative residuated lattice if

- $\langle A, \wedge, \vee, \otimes, 1 \rangle$ is a commutative lattice-ordered monoid.
- $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$.

Definition 2 (BL-algebras, BL-chains). A $\mathbf{BL}$ -algebra is a bounded, commutative residuated lattice which satisfies the divisibility property: if $x \leq y$ then $y \otimes (y \rightarrow x) = x$; pre-linearity: $(x \rightarrow y) \vee (y \rightarrow x) = 1$; bounded from below by $\perp$, i.e. $\perp \leq x$ for all $x \in A$, and integral in that 1 is the top element of the lattice, i.e. $x \leq 1$ for all $x \in A$. In this case we also denote 1 by $\top$. Finally, we note the condition of the divisibility property is equivalent to requiring that the residuated lattice satisfies the equation $x \otimes (x \rightarrow y) = y \otimes (y \rightarrow x) = x \wedge y$. A $\mathbf{BL}$ -chain is a totally-ordered $\mathbf{BL}$ -algebra.

Definition 3 (Monadic BL-algebras). $\mathbb{A} = \langle A, \vee, \wedge, \otimes, \rightarrow, \exists, \forall, \perp, \top \rangle$ is called a monadic BL-algebra (an **MBL**-algebra for short) if $\langle A, \vee, \wedge, \otimes, \rightarrow, \perp, \top \rangle$ is a **BL**-algebra, $\forall$ and $\exists$ are unary operations and the following identities are satisfied:

- (M1) $\forall x \rightarrow x = \top$.
- (M2) $\forall(x \rightarrow \forall y) = \exists x \rightarrow \forall y$.
- (M3) $\forall(\forall x \rightarrow y) = \forall x \rightarrow \forall y$.
- (M4) $\forall(\exists x \vee y) = \exists x \vee \forall y$.
- (M5) $\exists(x \otimes x) = \exists x \otimes \exists x$.

For brevity, if $\mathbb{A}$ is a **BL**-algebra and we enrich it with a monadic structure, we denote the resulting algebra by $\langle \mathbb{A}, \exists, \forall \rangle$. We denote by **MBL** the variety of **MBL**-algebras.

If in addition a **MBL**-algebra satisfies the chain condition:

$$(Chain) \quad \forall(x \vee y) = \forall x \vee \forall y,$$

then it is a **MBL**-chain. The variety of **MBL**-chains is denoted as **MBLC**.

Definition 4 (Hoop-algebra). A **Hoop**-algebra $\mathbb{A} = (A, \otimes, \rightarrow, 1)$ of type $(2, 2, 0)$ such that:

1. $(A, \otimes, 1)$ is a commutative monoid.
2. $x \rightarrow x = 1$.
3. $(x \otimes y) \rightarrow z = x \rightarrow (y \rightarrow z)$.
4. $x \otimes (x \rightarrow y) = y \otimes (y \rightarrow x)$ for all $x, y, z \in A$.

Definition 5 (Wajsberg-hoop). We say that a **Hoop**-algebra $\mathbb{A}$ is a **Wajsberg**-hoop if $\mathbb{A}$ satisfies:

1. $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ for all $x, y \in A$.

In **Wajsberg**-hoops, we can define $\wedge$ and $\vee$ as follows:

- $x \wedge y := x \otimes (x \rightarrow y)$.
- $x \vee y := (x \rightarrow y) \rightarrow y$.

A **Wajsberg**-hoop is a **Wajsberg**-chain if it is linearly ordered.

4 Validity in MBLC

Definition 6 (Denotation functions). Given a **MBLC**-algebra $\mathbb{A}_{\text{MBLC}}$, and an assignment from atomic formulas to elements of $\mathbb{A}_{\text{MBLC}}$:

$$Px \mapsto \llbracket Px \rrbracket \in \mathbb{A}_{\text{MBLC}}$$

We thus refer to the denotation of an atomic formula Px as $\llbracket Px \rrbracket_{\text{MBLC}}$.

We can extend that mapping to all formulas in the language of $\mathbb{L}_{\otimes MFO}$ in a straightforward way:

$$\begin{aligned}
\llbracket \phi \otimes \psi \rrbracket_{\mathbf{MBLC}} &:= \llbracket \phi \rrbracket_{\mathbf{MBLC}} \otimes \llbracket \psi \rrbracket_{\mathbf{MBLC}} \\
\llbracket \phi \wedge \psi \rrbracket_{\mathbf{MBLC}} &:= \llbracket \phi \rrbracket_{\mathbf{MBLC}} \wedge \llbracket \psi \rrbracket_{\mathbf{MBLC}} \\
\llbracket \phi \vee \psi \rrbracket_{\mathbf{MBLC}} &:= \llbracket \phi \rrbracket_{\mathbf{MBLC}} \vee \llbracket \psi \rrbracket_{\mathbf{MBLC}} \\
\llbracket \phi \rightarrow \psi \rrbracket_{\mathbf{MBLC}} &:= \llbracket \phi \rrbracket_{\mathbf{MBLC}} \rightarrow \llbracket \psi \rrbracket_{\mathbf{MBLC}} \\
\llbracket \forall x \phi \rrbracket_{\mathbf{MBLC}} &:= \forall \llbracket \phi \rrbracket_{\mathbf{MBLC}} \\
\llbracket \exists x \phi \rrbracket_{\mathbf{MBLC}} &:= \exists \llbracket \phi \rrbracket_{\mathbf{MBLC}}
\end{aligned}$$

Definition 7 (Validity). A formula ϕ is valid in a **MBLC**-algebra if $\top \leq \llbracket \psi \rrbracket$ holds under all assignments in this **MBLC**-algebra. A formula is said to be **MBLC**-valid if it is valid in all **MBLC**-algebras (notation: $\models_{\mathbf{MBLC}} \phi$). The valid formulas, in the sense above, are precisely the ones provable in monadic first-order basic logic of chains [7]:

Proposition 1. $\vdash_{\mathbf{MBLC}} \phi$ iff $\models_{\mathbf{MBLC}} \phi$.

Proof. See [7].

5 Kripke Semantics for MBLC

Note 2. The Kripke semantics for **MBLC** that we propose is something of a variant of our semantics introduced in [10] based on [8] [1], but is more immediately derived from [2]. We first need to define a particular class of functions from the set of worlds W to **Wajsberg**-chains attached with a bottom from below.

Definition 8 (Sloping functions). Let $\{\mathbb{A}_w\}_{w \in W}$ be a W -indexed family of **Wajsberg**-chains with almost disjoint domains except that they share the top element $\top$. We add the same new bottom element $\star$ to each $\mathbb{A}_w$ and get $\mathbb{A}_w \cup \{\star\}$. Let $\mathbb{W} = \langle W, \succeq \rangle$ be a linear order. A function $f: W \rightarrow \bigcup_{w \in W} (\mathbb{A}_w \cup \{\star\})$ is said to be a sloping function for **MBLC** (hereon sloping function, or sloping) if it is the $\top$ -constant function or there are $w_f \in W$ and $a_f \in \mathbb{A}_{w_f} - \{\top\}$ such that

$$f(w) = \begin{cases} \top & \text{if } w \succ w_f \\ a_f & \text{if } w = w_f \\ \star & \text{if } w \prec w_f \end{cases}$$

Lemma 1. *If $f, g: W \rightarrow \bigcup_{w \in W} (\mathbb{A}_w \cup \{\star\})$ are sloping, then so are the following functions⁴:*

$$\begin{aligned}
(f \wedge g)(w) &:= f(w) \wedge_{\mathbb{A}_w \cup \{\star\}} g(w) \\
(f \vee g)(w) &:= f(w) \vee_{\mathbb{A}_w \cup \{\star\}} g(w) \\
(f \otimes g)(w) &:= f(w) \otimes_{\mathbb{A}_w \cup \{\star\}} g(w) \\
(f \rightarrow g)(w) &:= \begin{cases} \top & \text{if } (w_f \succ w_g) \text{ or } (w_f = w_g \text{ and } a_f \leq a_g) \text{ or} \\ & \text{ } g \text{ is the } \top\text{-constant function} \\ g(w) & \text{if } w_f \prec w_g \text{ or } f \text{ is the } \top\text{-constant function} \\ \top & \text{if } w \succ w_f = w_g \text{ and } a_f > a_g \\ a_f \rightarrow_{\mathbb{A}_w} a_g & \text{if } w = w_f = w_g \text{ and } a_f > a_g \\ \star & \text{if } w \prec w_f = w_g \text{ and } a_f > a_g \end{cases}
\end{aligned}$$

where $\wedge, \vee, \otimes$ are extended to $\star$ such that $\star \wedge a = \star$, $\star \vee a = a$ and $\star \otimes a = \star$ for all $a \in \mathbb{A}_w \cup \{\star\}$.

Proof. The proof is similar to its counterpart in [10]. Let f, g be sloping functions. Let us consider each case:

– $f \wedge g$. When there are $\top$ -constant functions, the proof is easy. Otherwise,

$$(w_{f \wedge g}, a_{f \wedge g}) := \begin{cases} (w_f, a_f) & \text{if } w_f \succ w_g \\ (w_f, a_f \wedge a_g) & \text{if } w_f = w_g \\ (w_g, a_g) & \text{if } w_f \prec w_g \end{cases}$$

– $f \vee g$. When there are $\top$ -constant functions, the proof is easy. Otherwise,

$$(w_{f \vee g}, a_{f \vee g}) := \begin{cases} (w_f, a_f) & \text{if } w_f \prec w_g \\ (w_f, a_f \vee a_g) & \text{if } w_f = w_g \\ (w_g, a_g) & \text{if } w_f \succ w_g \end{cases}$$

– $f \otimes g$. When there are $\top$ -constant functions, the proof is easy. Otherwise,

$$(w_{f \otimes g}, a_{f \otimes g}) := \begin{cases} (w_f, a_f) & \text{if } w_f \succ w_g \\ (w_f, a_f \otimes a_g) & \text{if } w_f = w_g \\ (w_g, a_g) & \text{if } w_f \prec w_g \end{cases}$$

– $f \rightarrow g$. When f or g is the $\top$ -constant function, it is obvious. Similarly for the cases where $w_f \neq w_g$ or $(w_f = w_g \text{ and } a_f \leq a_g)$. When $w_f = w_g$ and $a_f > a_g$, then $w_{f \rightarrow g} = w_f = w_g$ and $a_{f \rightarrow g} = a_f \rightarrow_{\mathbb{A}_w} a_g$ (it is easy to see that $a_f \rightarrow_{\mathbb{A}_w} a_g < \top$).

In order to interpret **MBLC**, we still need the facilities to interpret the quantifiers. Indeed, we require that the underlying linear order to have two additional functions on it:

⁴ Here $\wedge_{\mathbb{A}_w \cup \{\star\}}$ means that the binary function corresponding to $\wedge$ is defined on $\mathbb{A}_w \cup \{\star\}$, and similarly for other functions indexed with **Wajsberg** chains.

Definition 9 (Monadic Heyting chain). A monadic Heyting chain is an algebra $\mathbb{H} = \langle H, \wedge, \vee, \rightarrow, \perp, \top, \forall, \exists \rangle$ such that $\langle H, \wedge, \vee, \rightarrow, \perp, \top \rangle$ is a linearly ordered Heyting algebra, and $\forall$ and $\exists$ satisfy the following conditions:

- (H1) $\forall a \leq a, a \leq \exists a$.
- (H2) $\forall(a \wedge b) = \forall a \wedge \forall b, \exists(a \vee b) = \exists a \vee \exists b$.
- (H3) $\forall \top = \top, \exists \perp = \perp$.
- (H4) $\forall \exists a = \exists a, \exists \forall a = \forall a$.
- (H5) $\exists(\exists a \wedge b) = \exists a \wedge \exists b$.

We define the order $\leq_{\mathbb{H}}$ on $\mathbb{H}$ such that $a \leq_{\mathbb{H}} b$ iff $a \wedge b = a$.

In what follows, we will use the elements of a monadic Heyting chain to be the ‘possible worlds’ of our relational semantics.

Definition 10 (CCVR-structure). Let $\mathbb{W} = \langle W, \succeq \rangle$ be the unique poset such that

- $W = \mathbb{H} - \{\top_{\mathbb{H}}\}$ where $\mathbb{H}$ is a monadic Heyting chain and $\top_{\mathbb{H}}$ is the top element of $\mathbb{H}$.
- For any $w, v \in W$, $w \preceq v$ iff $w \geq_{\mathbb{H}} v$.

That is to say, the order of $\mathbb{W}$ is the reverse order of that of $\mathbb{H} - \{\top_{\mathbb{H}}\}$. In what follows, we use $\leq$ and $\geq$ to represent the order on algebras and $\preceq$ and $\succeq$ to represent the order on $\mathbb{W}$. Since $\mathbb{H}$ is a linear order, the Heyting algebra part is defined in the expected way. We denote $\forall \mathbb{H} := \{\forall i \mid i \in \mathbb{H}\}$. We also assume the following conditions when labelling all $w \in W$ with **Wajsberg-chains**:

- If $\forall i < i$, $\mathbb{A}_{\forall i}$ has a second largest element $u_{\forall i}$.
- If $i < \exists i < \top_{\mathbb{H}}$, $\mathbb{A}_{\exists i}$ has a least element $0_{\exists i}$.
- In the top element w_1 of $\mathbb{W}$, $\mathbb{A}_{w_1}$ has a bottom element.

Let $\{\mathbb{A}_w\}_{w \in W}$ be a W -indexed family of **Wajsberg-chains** as described in Definition 8. A Castaño-Cimadamore-Varela-Rueda structure for $\{\mathbb{A}_w\}_{w \in W}$ (or **CCVR-structure**) is a pair $\mathbb{M} = \langle \mathbb{W}, \Vdash^{\text{CCVR}} \rangle$ where $\Vdash^{\text{CCVR}}$ is an infix operator (on worlds and propositional variables) taking values in $\bigcup_{w \in W} (\mathbb{A}_w \cup \{\star\})$, i.e. $(w \Vdash^{\text{CCVR}} p) \in \mathbb{A}_w \cup \{\star\}$, such that for any propositional variable p the function $\lambda w.(w \Vdash^{\text{CCVR}} p): W \rightarrow \bigcup_{w \in W} (\mathbb{A}_w \cup \{\star\})$ is a sloping function.

Definition 11 (CCVR Kripke Semantics for $\mathbb{L}_{\otimes}$). Given a **CCVR-structure**

$$\mathbb{M} = \langle \mathbb{W}, \Vdash^{\text{CCVR}} \rangle$$

the valuation function $w \Vdash^{\text{CCVR}} p$ on propositional variables p can be extended to all $\mathbb{L}_\otimes$ -formulas as:

$$\begin{aligned}
w \Vdash^{\text{CCVR}} \perp &:= \perp \\
w \Vdash^{\text{CCVR}} \phi \wedge \psi &:= (w \Vdash^{\text{CCVR}} \phi) \wedge_{\mathbb{A}_w \cup \{\star\}} (w \Vdash^{\text{CCVR}} \psi) \\
w \Vdash^{\text{CCVR}} \phi \vee \psi &:= (w \Vdash^{\text{CCVR}} \phi) \vee_{\mathbb{A}_w \cup \{\star\}} (w \Vdash^{\text{CCVR}} \psi) \\
w \Vdash^{\text{CCVR}} \phi \otimes \psi &:= (w \Vdash^{\text{CCVR}} \phi) \otimes_{\mathbb{A}_w \cup \{\star\}} (w \Vdash^{\text{CCVR}} \psi) \\
w \Vdash^{\text{CCVR}} \phi \rightarrow \psi &:= \begin{cases} \top & \text{if } w_\phi \succ w_\psi \text{ or } (w_f = w_g \text{ and } a_f \leq a_g) \text{ or} \\ & \lambda w. (w \Vdash^{\text{CCVR}} \psi) \text{ is the } \top\text{-constant function} \\ w \Vdash^{\text{CCVR}} \psi & \text{if } w_\phi \prec w_\psi \text{ or} \\ & \lambda w. (w \Vdash^{\text{CCVR}} \phi) \text{ is the } \top\text{-constant function} \\ \top & \text{if } w \succ w_\phi = w_\psi \text{ and } a_f > a_g \\ (w \Vdash^{\text{CCVR}} \phi) \rightarrow_{\mathbb{A}_w} (w \Vdash^{\text{CCVR}} \psi) & \text{if } w = w_\phi = w_\psi \text{ and } a_f > a_g \\ \star & \text{if } w \prec w_\phi = w_\psi \text{ and } a_f > a_g \end{cases} \\
w \Vdash^{\text{CCVR}} \forall x \phi &:= \begin{cases} w \Vdash^{\text{CCVR}} \phi \text{ if } w_\phi \in \forall \mathbb{H} \text{ or } \lambda w. (w \Vdash^{\text{CCVR}} \phi) \text{ is } \top\text{-constant} \\ \top & \text{if } w_\phi \notin \forall \mathbb{H} \text{ and } w \succ \forall w_\phi \\ \star & \text{if } w_\phi \notin \forall \mathbb{H} \text{ and } w \prec \forall w_\phi \\ u_{\forall i} & \text{if } w_\phi \notin \forall \mathbb{H} \text{ and } w = \forall w_\phi \end{cases} \\
w \Vdash^{\text{CCVR}} \exists x \phi &:= \begin{cases} w \Vdash^{\text{CCVR}} \phi \text{ if } w_\phi \in \exists \mathbb{H} \text{ or } \lambda w. (w \Vdash^{\text{CCVR}} \phi) \text{ is } \top\text{-constant} \\ \top & \text{if } w_\phi \notin \exists \mathbb{H} \text{ and } (w \succ \exists w_\phi \neq \top_{\mathbb{H}} \text{ or } \exists w_\phi = \top_{\mathbb{H}}) \\ \star & \text{if } w_\phi \notin \exists \mathbb{H} \text{ and } w \prec \exists w_\phi \neq \top_{\mathbb{H}} \\ 0_{\exists i} & \text{if } w_\phi \notin \exists \mathbb{H} \text{ and } w = \exists w_\phi \neq \top_{\mathbb{H}} \end{cases}
\end{aligned}$$

Lemma 2. For any formula ϕ the function $\lambda w. (w \Vdash^{\text{CCVR}} \phi) : W \rightarrow \bigcup_{w \in W} (\mathbb{A}_w \cup \{\star\})$ is a sloping function.

Proof. By induction on ϕ . The cases for $\psi \vee \xi, \psi \wedge \xi, \psi \otimes \xi$ and $\psi \rightarrow \xi$ follow directly from Definition 11 and Lemma 1. For the quantifiers cases, it is easy to see that the relevant function is a sloping function.

Lemma 3. (The sloping functions are linearly ordered in CCVR's.) Let $f, g : W \rightarrow \bigcup_{w \in W} (\mathbb{A}_w \cup \{\star\})$ be sloping for **MBLC**. Then:

$$\forall v : (f(v) \geq g(v)) \vee \forall v : (g(v) \geq f(v)).$$

Proof. It suffices to see that $\forall v : (f(v) \geq g(v))$ iff f is the $\top$ -constant function or $w_f \prec w_g$ or $(w_f = w_g \text{ and } a_f \geq a_g)$.

Corollary 1 (Monotonicity). The following (generalised) monotonicity property holds for all $\mathbb{L}_\otimes$ -formulas ϕ , i.e.

$$\text{if } w \preceq v \text{ then } (w \Vdash^{\text{CCVR}} \phi) \leq (v \Vdash^{\text{CCVR}} \phi).$$

Proof. This follows from the observation that the valuations are sloping functions, which are in turn monotonic functions.

In what follows we define the validity concepts in **CCVR**-structures.

Definition 12. – We say that a formula ϕ holds in a **CCVR**-structure $\mathbb{M}$ (written $\Vdash_{\mathbb{M}}^{\text{CCVR}} \phi$) if for all $w \in W$ we have

$$(w \Vdash^{\text{CCVR}} \phi) = \top$$

Otherwise, we say that the formula fails $\mathbb{M}$ (written $\nVdash_{\mathbb{M}}^{\text{CCVR}} \phi$) and this means:

$$\exists w \in W : (w \Vdash^{\text{CCVR}} \phi) < \top$$

– A formula ϕ is said to be valid under the **CCVR**-Kripke semantics (written $\Vdash^{\text{CCVR}} \phi$) if $\Vdash_{\mathbb{M}}^{\text{CCVR}} \phi$ for all **CCVR**-structures $\mathbb{M}$.

6 CCVR structures and ordinal sums of monadic BL chains

In what follows, we show the connection between **CCVR**-structures and ordinal sums of monadic BL chains.

Lemma 4. Given a **CCVR**-structure

$$\mathbb{M} = \langle \mathbb{W}, \Vdash^{\text{CCVR}} \rangle$$

and the sloping functions $\text{Slop}(\mathbb{M}) := \{f \mid f: W \rightarrow \bigcup_{w \in W} (\mathbb{A}_w \cup \{\star\}) \text{ is sloping}\}$ of $\mathbb{M}$, there is an isomorphism from $\text{Slop}(\mathbb{M})$ to $C_{\mathbb{H}} := \{(w, a) : w \in \mathbb{H} - \{\top_{\mathbb{H}}\}, a \in \mathbb{A}_w - \{\top_{\mathbb{A}_w}\}\} \cup \{(\top_{\mathbb{H}}, \top)\}$ (here $\top$ is the shared top-element of all algebras $\mathbb{A}_w$) such that the total order on $C_{\mathbb{H}}$ is defined as the following lexicographic order:

$$(i, a) \leq (j, b) \quad \text{iff} \quad i < j \quad \text{or} \quad (i = j \quad \text{and} \quad a \leq b)$$

We define on $C_{\mathbb{H}}$ the following operations:

$$(i, a) \wedge (j, b) = \min\{(i, a), (j, b)\}$$

$$(i, a) \vee (j, b) = \max\{(i, a), (j, b)\}$$

$$(i, a) \otimes (j, b) = \begin{cases} (i, a) & \text{if } i < j \\ (i, a \otimes b) & \text{if } i = j \\ (j, b) & \text{if } i > j \end{cases}$$

$$(i, a) \rightarrow (j, b) = \begin{cases} (1, 1) & \text{if } (i, a) \leq (j, b) \\ (i, a \rightarrow b) & \text{if } i = j \quad \text{and} \quad a > b \\ (j, b) & \text{if } i > j \end{cases}$$

$$\forall(i, a) = \begin{cases} (i, a) & \text{if } i \in \forall\mathbb{H} \\ (\forall i, u_{\forall i}) & \text{if } i \notin \forall\mathbb{H} \end{cases}$$

$$\exists(i, a) = \begin{cases} (i, a) & \text{if } i \in \forall\mathbb{H} \\ (\exists i, 0_{\exists i}) & \text{if } i \notin \forall\mathbb{H} \text{ and } \exists i \neq \top_{\mathbb{H}} \\ (\top_{\mathbb{H}}, \top) & \text{if } i \notin \forall\mathbb{H} \text{ and } \exists i = \top_{\mathbb{H}} \end{cases}$$

The operations $\wedge, \vee, \otimes, \rightarrow$ in $\text{Slop}(\mathbb{M}) := \{f: W \rightarrow \bigcup_{w \in W} (\mathbb{A}_w \cup \{\star\})\}$ are defined as in Lemma 1. The quantifiers in $\text{Slop}(\mathbb{M}) := \{f: W \rightarrow \bigcup_{w \in W} (\mathbb{A}_w \cup \{\star\})\}$ are defined as follows:

$$\begin{aligned} (\forall f)(w) &:= \begin{cases} f(w) & \text{if } w_f \in \forall\mathbb{H} \text{ or } f \text{ is the } \top\text{-constant function} \\ \top & \text{if } w_f \notin \forall\mathbb{H} \text{ and } w \succ \forall w_f \\ \star & \text{if } w_f \notin \forall\mathbb{H} \text{ and } w \prec \forall w_f \\ w_{\forall w_f} & \text{if } w_f \notin \forall\mathbb{H} \text{ and } w = \forall w_f \end{cases} \\ (\exists f)(w) &:= \begin{cases} f(w) & \text{if } w_f \in \exists\mathbb{H} \text{ or } f \text{ is the } \top\text{-constant function} \\ \top & \text{if } w_f \notin \exists\mathbb{H} \text{ and } (w \succ \exists w_f \neq \top_{\mathbb{H}} \text{ or } \exists w_{\phi} = \top_{\mathbb{H}}) \\ \star & \text{if } w_f \notin \exists\mathbb{H} \text{ and } w \prec \exists w_f \neq \top_{\mathbb{H}} \\ 0_{\exists w_f} & \text{if } w_f \notin \exists\mathbb{H} \text{ and } w = \exists w_f \neq \top_{\mathbb{H}} \end{cases} \end{aligned}$$

Proof. We map each $\top$ -constant function to $(\top_{\mathbb{H}}, \top)$ and other sloping functions to (w_f, a_f) . We denote this map as h . It is easy to see that h is well-defined and is a bijection. To show that h is an isomorphism, it suffices to show that h preserves operations, which is shown as follows:

- For the case of $\wedge$, let us consider $f, g \in \text{Slop}(\mathbb{M})$.
 - If both f and g are $\top$ -constant functions, then $h(f) = h(g) = (\top_{\mathbb{H}}, \top)$, and so $f \wedge g = g$ and $(\top_{\mathbb{H}}, \top) \wedge (\top_{\mathbb{H}}, \top) = (\top_{\mathbb{H}}, \top \wedge \top) = (\top_{\mathbb{H}}, \top)$. Therefore, $h(f \wedge g) = h(g) = (\top_{\mathbb{H}}, \top) = (\top_{\mathbb{H}}, \top) \wedge (\top_{\mathbb{H}}, \top) = h(f) \wedge h(g)$.
 - If f is the $\top$ -constant function and g is not, then $f \wedge g = g$, $h(f) = (\top_{\mathbb{H}}, \top)$ and $h(g) = (w, a)$ for some $w \in \mathbb{H} - \{\top_{\mathbb{H}}\}$ and $a \in \mathbb{A}_w - \{\top_{\mathbb{A}_w}\}$. Since $w <_{\mathbb{H}} \top_{\mathbb{H}}$, we have that $(\top_{\mathbb{H}}, \top) \wedge (w, a) = (w, a)$. Then $h(f \wedge g) = h(g) = (w, a) = (\top_{\mathbb{H}}, \top) \wedge (w, a) = h(f) \wedge h(g)$.
 - If g is the $\top$ -constant function and f is not, the proof is similar.
 - If neither f nor g are $\top$ -constant functions, then there are $w_1, w_2 \in \mathbb{H} - \{\top_{\mathbb{H}}\}$ and $a_1, a_2 \in \mathbb{A}_w - \{\top_{\mathbb{A}_w}\}$ such that $h(f) = (w_1, a_1)$ and $h(g) = (w_2, a_2)$. Then $h(f) \wedge h(g) = (w_1, a_1) \wedge (w_2, a_2)$.
 - * If $w_1 = w_2$, then $h(f) \wedge h(g) = (w_1, a_1 \wedge a_2)$, and

$$(f \wedge g)(w) := \begin{cases} \star & \text{if } w \prec w_1 \\ a_1 \wedge a_2 & \text{if } w = w_1 \\ \top & \text{if } w \succ w_1 \end{cases}$$
 so $h(f \wedge g) = (w_1, a_1 \wedge a_2) = h(f) \wedge h(g)$.

- * If $w_1 \neq w_2$, without loss of generality we assume that $w_1 \prec w_2$, then $h(f) \wedge h(g) = (w_1, a_1) \wedge (w_2, a_2) = (w_2, a_2)$, and

$$(f \wedge g)(w) := \begin{cases} \star \wedge \star = \star & \text{if } w \prec w_1 \\ a_1 \wedge \star = \star & \text{if } w = w_1 \\ \top \wedge \star = \star & \text{if } w_1 \prec w \prec w_2 \\ \top \wedge a_2 = a_2 & \text{if } w = w_2 \\ \top \wedge \top = \top & \text{if } w \succ w_2 \end{cases}$$

therefore $h(f \wedge g) = (w_2, a_2) = h(f) \wedge h(g)$.

Therefore in both cases we have $h(f \wedge g) = h(f) \wedge h(g)$.

- The case of $\vee$ is similar to the case of $\wedge$.

- For the case of $\otimes$, let us consider $f, g \in \text{Slop}(\mathbb{M})$.

- If both f and g are $\top$ -constant functions, then $h(f) = h(g) = (\top_{\mathbb{H}}, \top)$, and it is easy to see that $f \otimes g = g$ and $(\top_{\mathbb{H}}, \top) \otimes (\top_{\mathbb{H}}, \top) = (\top_{\mathbb{H}}, \top \otimes \top) = (\top_{\mathbb{H}}, \top)$. Therefore, $h(f \otimes g) = h(g) = (\top_{\mathbb{H}}, \top) = (\top_{\mathbb{H}}, \top) \otimes (\top_{\mathbb{H}}, \top) = h(f) \otimes h(g)$.
- If f is the $\top$ -constant function and g is not, then $f \otimes g = g$, $h(f) = (\top_{\mathbb{H}}, \top)$ and $h(g) = (w, a)$ for some $w \in \mathbb{H} - \{\top_{\mathbb{H}}\}$ and $a \in \mathbb{A}_w - \{\top_{\mathbb{A}_w}\}$. Since $w <_{\mathbb{H}} \top_{\mathbb{H}}$, we have that $(\top_{\mathbb{H}}, \top) \otimes (w, a) = (w, a)$. Then $h(f \otimes g) = h(g) = (w, a) = (\top_{\mathbb{H}}, \top) \otimes (w, a) = h(f) \otimes h(g)$.
- If g is the $\top$ -constant function and f is not, the proof is similar.
- If neither f nor g are $\top$ -constant functions, then there are $w_1, w_2 \in \mathbb{H} - \{\top_{\mathbb{H}}\}$ and $a_1, a_2 \in \mathbb{A}_w - \{\top_{\mathbb{A}_w}\}$ such that $h(f) = (w_1, a_1)$ and $h(g) = (w_2, a_2)$. Then $h(f) \otimes h(g) = (w_1, a_1) \otimes (w_2, a_2)$.
 - * If $w_1 = w_2$, then $h(f) \otimes h(g) = (w_1, a_1 \otimes a_2)$, and

$$(f \otimes g)(w) := \begin{cases} \star & \text{if } w \prec w_1 \\ a_1 \otimes a_2 & \text{if } w = w_1 \\ \top & \text{if } w \succ w_1 \end{cases}$$

so $h(f \otimes g) = (w_1, a_1 \otimes a_2) = h(f) \otimes h(g)$.

- * If $w_1 \neq w_2$, without loss of generality we assume that $w_1 \prec w_2$, then $h(f) \otimes h(g) = (w_1, a_1) \otimes (w_2, a_2) = (w_2, a_2)$, and

$$(f \otimes g)(w) := \begin{cases} \star \otimes \star = \star & \text{if } w \prec w_1 \\ a_1 \otimes \star = \star & \text{if } w = w_1 \\ \top \otimes \star = \star & \text{if } w_1 \prec w \prec w_2 \\ \top \otimes a_2 = a_2 & \text{if } w = w_2 \\ \top \otimes \top = \top & \text{if } w \succ w_2 \end{cases}$$

therefore $h(f \otimes g) = (w_2, a_2) = h(f) \otimes h(g)$.

Therefore in both cases we have $h(f \otimes g) = h(f) \otimes h(g)$.

- For the case of $\rightarrow$, let us consider $f, g \in \text{Slop}(\mathbb{M})$.

- If both f and g are $\top$ -constant functions, then $f \rightarrow g$ is also the $\top$ -constant function, so $h(f \rightarrow g) = h(f) = h(g) = (\top_{\mathbb{H}}, \top)$. Since $(\top_{\mathbb{H}}, \top) \rightarrow (\top_{\mathbb{H}}, \top) = (\top_{\mathbb{H}}, \top)$, we have $h(f \rightarrow g) = (\top_{\mathbb{H}}, \top) = (\top_{\mathbb{H}}, \top) \rightarrow (\top_{\mathbb{H}}, \top) = h(f) \rightarrow h(g)$.

- If f is the $\top$ -constant function and g is not, then $f \rightarrow g = g$, $h(f) = (\top_{\mathbb{H}}, \top)$ and $h(g) = (w, a)$ for some $w \in \mathbb{H} - \{\top_{\mathbb{H}}\}$ and $a \in \mathbb{A}_w - \{\top_{\mathbb{A}_w}\}$. Since $w <_{\mathbb{H}} \top_{\mathbb{H}}$, we have that $(\top_{\mathbb{H}}, \top) \rightarrow (w, a) = (w, a)$. Then $h(f \rightarrow g) = h(g) = (w, a) = (\top_{\mathbb{H}}, \top) \rightarrow (w, a) = h(f) \rightarrow h(g)$.
 - If g is the $\top$ -constant function and f is not, then $f \rightarrow g = g$ is the $\top$ -constant function, $h(g) = (\top_{\mathbb{H}}, \top)$ and $h(f) = (w, a)$ for some $w \in \mathbb{H} - \{\top_{\mathbb{H}}\}$ and $a \in \mathbb{A}_w - \{\top_{\mathbb{A}_w}\}$. Since $w <_{\mathbb{H}} \top_{\mathbb{H}}$, we have that $(w, a) \rightarrow (\top_{\mathbb{H}}, \top) = (\top_{\mathbb{H}}, \top)$. Then $h(f \rightarrow g) = h(g) = (\top_{\mathbb{H}}, \top) = (w, a) \rightarrow (\top_{\mathbb{H}}, \top) = h(f) \rightarrow h(g)$.
 - If neither f nor g are $\top$ -constant functions, then there are $w_1, w_2 \in \mathbb{H} - \{\top_{\mathbb{H}}\}$ and $a_1, a_2 \in \mathbb{A}_w - \{\top_{\mathbb{A}_w}\}$ such that $h(f) = (w_1, a_1)$ and $h(g) = (w_2, a_2)$. Then $h(f) \rightarrow h(g) = (w_1, a_1) \rightarrow (w_2, a_2)$.
 - * If $w_1 = w_2$ and $a_1 \leq a_2$, then $h(f) \rightarrow h(g) = (w_1, a_1) \rightarrow (w_2, a_2) = (\top_{\mathbb{H}}, \top)$, and $f \rightarrow g$ is the $\top$ -constant function, so $h(f \rightarrow g) = (\top_{\mathbb{H}}, \top) = (w_1, a_1) \rightarrow (w_2, a_2) = h(f) \rightarrow h(g)$.
 - * If $w_1 = w_2$ and $a_1 > a_2$, then $h(f) \rightarrow h(g) = (w_1, a_1) \rightarrow (w_2, a_2) = (w_1, a_1 \rightarrow a_2)$, and

$$(f \rightarrow g)(w) := \begin{cases} \perp & \text{if } w < w_1 \\ a_1 \rightarrow a_2 & \text{if } w = w_1 \\ \top & \text{if } w > w_1 \end{cases}$$
 so $h(f \rightarrow g) = (w_1, a_1 \rightarrow a_2) = h(f) \rightarrow h(g)$.
 - * If $w_1 < w_2$, then $h(f) \rightarrow h(g) = (w_1, a_1) \rightarrow (w_2, a_2) = (w_2, a_2)$, and $f \rightarrow g = g$, therefore $h(f \rightarrow g) = h(g) = (w_2, a_2) = (w_1, a_1) \rightarrow (w_2, a_2) = h(f) \rightarrow h(g)$.
 - * If $w_1 > w_2$, then $h(f) \rightarrow h(g) = (w_1, a_1) \rightarrow (w_2, a_2) = (\top_{\mathbb{H}}, \top)$, and $f \rightarrow g$ is the $\top$ -constant function, so $h(f \rightarrow g) = (\top_{\mathbb{H}}, \top) = (w_1, a_1) \rightarrow (w_2, a_2) = h(f) \rightarrow h(g)$.
- Therefore in all four cases we have $h(f \rightarrow g) = h(f) \rightarrow h(g)$.

- For the case of $\exists$, let us consider $f \in \text{Slop}(\mathbb{M})$.
 - If f is the $\top$ -constant function, then $h(f) = (\top_{\mathbb{H}}, \top)$, and $\exists f$ is also the $\top$ -constant function, so $h(\exists f) = (\top_{\mathbb{H}}, \top)$. Since $\top_{\mathbb{H}} = \forall \top_{\mathbb{H}}$, we have that $\top_{\mathbb{H}} \in \forall \mathbb{H}$, so $h(\exists f) = (\top_{\mathbb{H}}, \top) = \exists(\top_{\mathbb{H}}, \top) = \exists h(f)$.
 - If f is not the $\top$ -constant function, then there are $w \in \mathbb{H} - \{\top_{\mathbb{H}}\}$ and $a \in \mathbb{A}_w - \{\top_{\mathbb{A}_w}\}$ such that $h(f) = (w_f, a_f)$. If $w_f \in \forall \mathbb{H}$, then $\exists f$ is the same as f , therefore $h(\exists f) = h(f) = (w_f, a_f) = \exists(w_f, a_f) = \exists h(f)$. Otherwise, $w_f \notin \forall \mathbb{H}$, and we discuss in two cases:
 - * If $\exists w_f = \top_{\mathbb{H}}$, then $\exists f$ is the $\top$ -constant function, and therefore $h(\exists f) = (\top_{\mathbb{H}}, \top) = (\exists w_f, \top) = \exists(w_f, a_f) = \exists h(f)$.
 - * If $\exists w_f \neq \top_{\mathbb{H}}$, then $h(\exists f) = (\exists w_f, 0_{\exists w_f}) = \exists(w_f, a_f) = \exists h(f)$.
- The case of $\forall$ is similar to the case of $\exists$.

7 Soundness and completeness of CCVR-semantics

We now prove the soundness of the Kripke semantics for **MBLC**.

Theorem 1 (Soundness). *If $\vdash_{\mathbf{MBLC}} \phi$ then $\Vdash^{\mathbf{CCVR}} \phi$.*

Proof. Suppose $\vdash \phi$ holds in **MBLC**. By the algebraic completeness result for **MBLC** algebras with respect to the Hilbert-style proof system **MBLC_H** (see Proposition 1), it follows that for all **MBLC**-algebras $\mathbb{G}$ and all mappings $h: \text{Atom} \rightarrow \mathbb{G}$ from atomic formulas to elements of $\mathbb{G}$, we have that $\llbracket \phi \rrbracket_h^{\mathbb{G}} = \top$.

Now for any **CCVR**-structure $\mathbb{M}$, by Lemma 4, $\text{Slop}(\mathbb{M})$ is isomorphic to some $C_{\mathbb{H}}$. By ([2, Theorem 6.1]), $C_{\mathbb{H}}$ is a monadic BL chain, therefore we have $C_{\mathbb{H}} \models \phi$, so $\mathbb{M} \Vdash \phi$.

We now prove completeness of **MBLC** for the semantics presented above. Our proof is a departure from our earlier papers [10] and [9]. In the former, we simply embed our semantics into the poset products of Jipsen and Montagna, while in the latter case we exploit properties of **BL**-algebra representability (both in terms of poset products and ordinal sums) unique to that setting. Here, the result follows immediately by the representation of **MBLC**-algebras by ordinal sums and our isomorphism result in Lemma 4.

Theorem 2 (Completeness). *If $\Vdash^{\mathbf{CCVR}} \phi$ then $\vdash_{\mathbf{MBLC}} \phi$.*

Proof. Suppose $\vdash \phi$ fails in **MBLC**. By the algebraic completeness result for **MBLC**-algebras with respect to the Hilbert-style proof system **MBLC_H** (see Proposition 1), it follows that for some **MBLC**-algebra $\mathbb{G}$ and some mapping $h: \text{Atom} \rightarrow \mathbb{G}$ from atomic formulas to elements of $\mathbb{G}$, we have that $\llbracket \phi \rrbracket_h^{\mathbb{G}} \neq \top$. By ([2, Theorem 6.2]) we can take $\mathbb{G}$ to be some $C_{\mathbb{H}}$ as defined in Lemma 4. By Lemma 4, this $C_{\mathbb{H}}$ is isomorphic to the sloping function algebra $\text{Slop}(\mathbb{M})$ of a **CCVR**-structure $\mathbb{M}$ where the valuation on $\mathbb{M}$ agrees with h so that $\mathbb{M} \not\models \phi$.

8 Conclusion

In the preceding, we extracted a relational semantics from the ordinal sum representation of [2] for **MBLC**. By introducing generalisations of Kripke semantics adequate for **MBLC** (and neighbours, see [9], [10] [3]), we seek a new perspective on fuzzy logics such as **MBLC** as constructive or semi-constructive systems. Moreover, relational semantics typically come pregnant with connections to proof theory, decidability and model theory. We hope in the fullness of time to exploit the semantics developed here to develop semantically motivated calculi **MBLC** and similar systems. We also submit the question of how deep the analogy with intuitionistic systems runs (via Kripke semantics) has independent theoretical value.

There are several alternatives towards this end. One approach to consider are semantic tableaux and refutation systems: One can exploit the notion of unsatisfiability in our relational semantics to devise a proof system, as is done in classical modal or intuitionistic logic (see [12]). Another possibility is to consider labelled calculi, see e.g. [11] or [4]. Labelled calculi have been well-tested in

cases of classical modal logics, extensions of classical modal systems, substructural cases, as well as logics over an intuitionistic or minimal base, although as yet untested on cases involving of divisibility ($[(\wedge 1)]$ and $[(\wedge 2)]$, page 2). Still another alternative are the multi-type calculi of e.g. [5]. These calculi seem well-adapted to systems defined over distributive structures, e.g. chains, having strong algebraic underpinnings.

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