Dimension prints and the avoidance of sets for flow solutions of non-autonomous ordinary differential equations

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Abstract

We provide a criterion for a generalised flow solution of a non-autonomous ordinary differential equation to avoid a subset of the phase space. This improves on that established by Aizenman for the autonomous case, where avoidance is guaranteed if the underlying vector field is sufficiently regular and the subset has sufficiently small box-counting dimension. We define the r-codimension print of a subset $S \subset \mathbb{R}^n \times [0,T]$, which is a subset of $(0,\infty]^2$ that encodes the dimension of S in a way that distinguishes spatial and temporal detail. We prove that the subset S is avoided by a generalised flow solution with underlying vector field in $L^p\left(0,T;L^q\left(\mathbb{R}^n\right)\right)$ with $1 \leq p,q \leq \infty$ if the Hölder conjugates (q^*,p^*) are in the r-codimension print of S.

Keywords: Generalised flow, DiPerna-Lions flow, irregular vector field, non-autonomous ordinary differential equation, dimension print, avoidance of sets

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1. Preliminaries

1.1. Ordinary differential equations

We examine the ordinary differential equation

$$\dot{x} = f(x, t) \tag{1}$$

where the vector field f is of limited regularity, typically $f \in L^p(0, T; L^q(\mathbb{R}^n))$ for some p, q with $1 \leq p, q \leq \infty$. Such equations arise naturally in fluid mechanics, for example: if f is a solution of the Navier-Stokes equations, which currently have limited known regularity, then solving (1) allows us to recover the trajectories of the fluid particles (see Foias et al. (1985)).

In this irregular setting the notion of a classical flow solution is too strong to be useful as the vector field may contain discontinuities in which case solutions of (1) that encounter these discontinuities will not be continuously differentiable. Further, the classical flow is not invariant under the equivalence classes of the L^p spaces, which is to say that if f is almost everywhere equal to g then a classical flow solution of (1) is not necessarily a classical flow solution of $\dot{x} = g(x, t)$. This invariance is desirable as it allows solutions of (1) to be found using functional analytic methods. The first general theory for ordinary differential equations with vector fields of limited regularity was described in the seminal paper of DiPerna and Lions (1989) in which the authors define a generalised flow solution that is invariant under the choice of representative for the vector field f.

Generalised flow solutions can be thought of as aggregates of individual trajectories of (1):

Definition 1.1. *A map $\xi\colon [0, T] \to \mathbb{R}^n$ is a* trajectory *of* (1) *with initial data $(x, s) \in \mathbb{R}^n \times [0, T]$ if ξ is absolutely continuous and*

$$\xi(t) = x + \int_s^t f(\xi(\tau), \tau)\, \mathrm{d}\tau \tag{2}$$

for all $t \in [0, T]$.

Note that as trajectories are absolutely continuous the equality (2) is equivalent to requiring that $\dot{\xi} = f(\xi(t), t)$ for almost every $t \in [0, T]$, and that, unlike the classical continuously differentiable solutions, we do not require $t \mapsto f(\xi(t), t)$ to be continuous.

A generalised flow is an aggregate of trajectories, one for almost all initial data, with some additional properties:

Definition 1.2. *A map $X\colon [0,T] \times \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is a generalised flow solution of (1) if*

- *for almost every $x \in \mathbb{R}^n$ and all $s \in [0,T]$ the map $t \mapsto X(t,x,s)$ is a trajectory of (1) with initial data (x,s),*
- *the map X satisfies the group property:*
$$X(t, X(\tau, x, s), \tau) = X(t, x, s) \qquad \forall t, s, \tau \in [0,T] \tag{3}$$
for almost every $x \in \mathbb{R}^n$, and
- *there exists a constant $C \in \mathbb{R}$ such that*
$$e^{-|t-s|C} \mathcal{L}^n(B) \leq \mathcal{L}^n\left(X(t,\cdot,s)^{-1} B\right) \leq e^{|t-s|C} \mathcal{L}^n(B) \tag{4}$$
for all $t, s \in [0,T]$ and all Borel sets $B \subset \mathbb{R}^n$,

where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure.

The group property (3) guarantees that almost every initial condition $(x,s) \in \mathbb{R}^n \times [0,T]$ lies on exactly one trajectory and further, as (3) implies that
$$X(t,x,s) = X(t, X(0,x,s), 0) \qquad \forall t, s \in [0,T] \tag{5}$$
for almost every $x \in \mathbb{R}^n$, that almost every trajectory $t \mapsto X(t,x,s)$ can be written as a trajectory with initial temporal data $s = 0$. We will see in the next section that this simplifies our approach to non-autonomous avoidance.

The constant (4) requires that the Lebesgue measure of spatial sets $B \subset \mathbb{R}^n$ does not change dramatically as the set is transported along the trajectories of the flow both forwards and backwards in time. In particular, null sets remain null and sets of positive measure continue to have positive measure as they are transported along trajectories. Note that in the classical case, for a smooth vector field f and a classical flow solution X, it is relatively straightforward to show that
$$\mathcal{L}^n\left(X(t,\cdot,s)^{-1} B\right) = \int_B \exp\left(\int_t^s \mathrm{div} f(X(\tau,x,t),\tau)\,\mathrm{d}\tau\right) \mathrm{d}x \tag{6}$$

for all $t, s \in [0,T]$ and all Borel sets $B \subset \mathbb{R}^n$. If additionally the divergence of f is bounded we see from (6) that (4) holds with $C := \|\mathrm{div} f\|_\infty$.

For autonomous vector fields $f\colon \mathbb{R}^n \to \mathbb{R}^n$ we adapt the above definitions by extending the temporal domain $[0,T]$ to the whole of $\mathbb{R}$, writing $f(x,t) := f(x)$ and noting that the trajectories are independent of the initial temporal data s. Consequently, we drop the dependence on s and the group property becomes

$$X(t+\tau, x) = X(t, X(\tau, x)) \qquad \forall t, \tau \in \mathbb{R} \tag{7}$$

for almost every $x \in \mathbb{R}^n$.

The importance of generalised flow solutions lies in the main theorem of DiPerna and Lions (1989) in which the authors demonstrate that such solutions exist and are unique under mild assumptions on the vector field f:

Theorem 1.3 (DiPerna and Lions (1989))**.** *There exists a unique generalised flow solution of* (1) *if the vector field f satisfies*

1) $f \in L^1\left(0,T; W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)\right)$,
2) $f/(1+|x|) \in L^1(0,T; L^1(\mathbb{R}^n)) + L^1(0,T; L^\infty(\mathbb{R}^n))$, *and*
3) *the distributional divergence* $\mathrm{div} f \in L^1(0,T; L^\infty(\mathbb{R}^n))$.

1.2. Avoidance

For a compact subset of the phase space $S \subset \mathbb{R}^n \times [0,T]$ we say that a trajectory avoids the set S if it does not intersect S at any time $t \in [0,T]$. We say that a generalised flow avoids a set S if almost all of its trajectories avoid S:

Definition 1.4. *A generalised flow solution $X\colon [0,T] \times \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ avoids a compact subset $S \subset \mathbb{R}^n \times [0,T]$ if the set*

$$\{x \in \mathbb{R}^n | \, (X(t,x,0), t) \in S \quad \textit{for some}\, t \in [0,T]\} \tag{8}$$

has zero n-dimensional Lebesgue measure.

In the case that $S = A \times [0,T]$ with $A \subset \mathbb{R}^n$ we can regard the subset S as a set of spatial points which are to be avoided at all times. In this case as $(X(t,x,\tau), t) \in S$ if and only if $X(t,x,\tau) \in A$ the above definition of avoidance reduces to that used in the current literature (Aizenman

(1978b); Cipriano and Cruzeiro (2005); Robinson and Sadowski (2009)) which only considers avoidance of sets of this form. The 'autonomous' avoidance property first appeared in Nelson (1962) (on pp.163) but was named and extensively studied in Aizenman (1978b).

The avoidance condition established in Aizenman (1978b) for the autonomous case is given only in terms of the regularity of the vector field f and the 'size' of the subset $A \subset \mathbb{R}^n$ (in the sense of the upper box-counting dimension, $\dim_B(A)$, which we recall below):

Theorem 1.5 (Aizenman (1978b))**.** *Let $f \in L^q(\mathbb{R}^n)$ and let X be a generalised flow solution of $\dot{x} = f(x)$. If a bounded subset $A \subset \mathbb{R}^n$ satisfies*

$$\frac{1}{q} + \frac{1}{n - \dim_B(A)} < 1 \tag{9}$$

then the generalised flow solution X avoids the set A.

In Cipriano and Cruzeiro (2005) this result is partially extended to the non-autonomous case: the same criterion (9) guarantees avoidance of product sets $S = A \times [0, T]$ for vector fields $f \in L^1(0, T; L^q(\mathbb{R}^n))$. In this paper we extend this result to the general case where $S \subset \mathbb{R}^n \times [0, T]$ is an arbitrary bounded subset and we know the regularity of the vector field with respect to both the spatial and temporal components in the sense that we know that $f \in L^p(0, T; L^q(\mathbb{R}^n))$ for some $1 \leq p, q \leq \infty$.

Using the result of Cipriano & Cruzeiro we can immediately establish a partial result for the general non-autonomous case: if $P_x \colon \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ is the canonical projection onto the spatial component then $S \subset P_x(S) \times [0, T]$ so avoidance of $P_x(S) \times [0, T]$ entails avoidance of S. Consequently, the result in Cipriano and Cruzeiro (2005) implies that the flow avoids the set S if

$$\frac{1}{q} + \frac{1}{n - \dim_B(P_x(S))} < 1. \tag{10}$$

This approach ignores the temporal regularity of f and the temporal detail of S in the sense that it does not distinguish between the subsets $A \times [0, T]$ and $A \times \{0\}$ of the phase space despite the fact that the latter set is smaller and intuitively feels 'more avoidable'. In Section 2 we provide a more general criterion than (10) that guarantees avoidance and takes into account both the spatial and temporal detail of the set S and the vector field f. To this end we introduce the r-codimension print, an extended notion of dimension similar to the Hausdorff dimension print of Rogers (1988), which encodes the detail of S appropriately for non-autonomous avoidance.

1.3. Box-counting dimension

We recall that the upper and lower box-counting dimensions of a bounded set $A \subset \mathbb{R}^n$ are given by

$$\dim_B(A) = \limsup_{\delta \to 0} \frac{\log N(A,\delta)}{-\log \delta}$$
$$\dim_{LB}(A) = \liminf_{\delta \to 0} \frac{\log N(A,\delta)}{-\log \delta}$$

respectively, where $N(A,\delta)$ is the smallest number of sets with diameter at most δ which form a cover of A, or one of many similar quantities which give an equivalent definition (discussed in Falconer (2003) §3.1 'Equivalent Definitions'). Throughout we take $0 < \delta < 1$ so that $-\log \delta$ is strictly positive. Here we use the alternative 'Minkowski sausage' formulation (again, see Falconer (2003) for proof of equivalence)

$$\dim_B(A) = n - \liminf_{\delta \to 0} \frac{\log(\mathcal{L}^n(A_\delta))}{\log \delta} \tag{11}$$

$$\dim_{LB}(A) = n - \limsup_{\delta \to 0} \frac{\log(\mathcal{L}^n(A_\delta))}{\log \delta} \tag{12}$$

where $A_\delta := \{x \in \mathbb{R}^n | \mathrm{dist}(x,A) < \delta\}$ is the δ-neighbourhood of A. Essentially, if $\mathcal{L}^n(A_\delta)$ scales like $\delta^{n-\varepsilon}$ as $\delta \to 0$ then the box-counting dimensions capture ε giving an indication of the growth of the δ-neighbourhood of A. In fact we have the following bounds on $\mathcal{L}^n(A_\delta)$:

Lemma 1.6. *Let A be a bounded, non-empty subset of $\mathbb{R}^n$. For each α and β such that $\alpha > n - \dim_{LB}(A)$ and $\beta < n - \dim_B(A)$ and each $\delta^* > 0$ there exists a constant C such that*

$$\frac{1}{C}\delta^\alpha \leq \mathcal{L}^n(A_\delta) \leq C\delta^\beta \qquad \forall \delta \in [0,\delta^*] \tag{13}$$

The growth of $\mathcal{L}^n(A_\delta)$ at small length scales reflects how 'spread out' the set is at these length scales: rapid growth as δ increases indicates that the δ neighbourhoods around a significant number of individual points of A do not intersect by a large amount.

1.4. Dimension prints

The box-counting dimension fails to capture some significant geometry of sets: if C is the Cantor 'middle half' set, which has Hausdorff and box-counting dimension equal to $\frac{1}{2}$, then the product set $C \times C \subset \mathbb{R}^2$ has Hausdorff and box-counting dimension equal to 1 (see Example 7.6 in Falconer (2003)). Consequently, $C \times C$ has the same Hausdorff and box-counting dimension as a line segment in $\mathbb{R}^2$ yet the sets have different anisotropic (i.e. directionally dependent) detail in the sense that the product set has detail in two directions while the line segment has detail in only one direction. One way of encoding this detail is in a 'dimension print', initially developed in Rogers (1988).

Recall that the Hausdorff measures $\mathcal{H}^d$ are a 1-parameter (in d) family of measures and that the Hausdorff dimension of a set S is the value of the parameter at which $\mathcal{H}^d(S)$ changes from infinity to zero. To capture the anisotropic properties of subsets of $\mathbb{R}^n$ in Rogers (1988) the author defines an n-parameter family of measures $\mathcal{H}^\alpha$ similar to the Hausdorff measures. The dimension print of a set S is the set of points α for which $\mathcal{H}^\alpha(S)$ is non-zero.

Definition 1.7 (Rogers (1988))**.** *For a subset $S \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}^n$ with $\alpha_j \geq 0$ for all j we define for all $\delta > 0$ the quantity*

$$\mathcal{H}^\alpha_\delta(S) := \inf\left\{\sum_{i=1}^{\infty} l_1(B_i)^{\alpha_1}\dots l_n(B_i)^{\alpha_n} \,\middle|\, B_i \in \mathcal{B},\ \mathrm{diam} B_i \leq \delta,\ \cup_{i=1}^{\infty} B_i \supset S\right\}$$

where $\mathcal{B}$ is the set of open rectangular parallelepipeds (henceforth 'boxes') in $\mathbb{R}^n$, $l_1(B_i), l_2(B_i), \dots, l_n(B_i)$ are the side lengths of the box B_i taken in a non-increasing order and $l_j(B_i)^0 = 1$ for all i, j.

We say that α is in the Hausdorff dimension print of S if and only if the Hausdorff-type measure

$$\mathcal{H}^\alpha(S) := \sup_{\delta>0} \mathcal{H}^\alpha_\delta(S)$$

is positive.

As each measure weights the side lengths of the boxes differently it is possible to distinguish between sets that are easily covered by long thin boxes, such as a line, and sets which are not, such as the product set $C \times C$. Note that the measure $\mathcal{H}^{(d,0,\dots,0)}$ is equal to the usual d-dimensional Hausdorff measure multiplied by a constant depending only on n, so it is possible to read the Hausdorff dimension of a set directly from the Hausdorff dimension print. Also note that we do not require the boxes B_i to have sides parallel to the coordinate axes so that the Hausdorff dimension print captures the degree to which a set has directionally dependent detail but not the direction in which this detail lies. In particular the dimension print is invariant under Euclidean transformations of a set as we can simply apply the same transformation to each of the covering boxes B_i. While this is generally regarded as a desirable property for any notion of 'dimension', we ultimately wish to distinguish between spatial detail and temporal detail when we consider the non-autonomous ODE (1).

At the expense of this Euclidean invariance we can use dimension prints to capture the direction in which the detail lies by instead restricting the class of boxes $\mathcal{B}$ in Definition 1.7 to be those with sides parallel to the coordinate axes and each $l_j(B_i)$ to be the length of the side of the box B_i which is parallel to the j^{th} coordinate axis.

In Lee and Baek (1995) a box-counting dimension print is defined in a similar way from the premeasure

$$\mu^{\alpha}(S) = \liminf_{\delta \to 0} \{N_l(S)\, l_1^{\alpha_1} \dots l_n^{\alpha_n} | 0 < l_1 \leq l_2 \leq \dots \leq l_n \leq \delta\}$$

where, after dividing $\mathbb{R}^n$ into mesh boxes with dimensions $l_1 \times l_2 \times \dots \times l_n$, the quantity $N_l(S)$ is the number of mesh boxes which intersect the set S. In the next section we define a similar print, which is useful for our study of non-autonomous avoidance.

1.5. r-codimension print

We introduce the r-codimension print of a bounded non-empty set $S \subset \mathbb{R}^n \times [0, T]$ as a way of encoding the anisotropic fractal detail of subsets appropriately for the study of non-autonomous avoidance. The r-codimension print takes its name from the function $\operatorname{dist}(\cdot, S) : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ used in its definition, which is denoted by r in Aizenman (1978b).

Definition 1.8. *The r-codimension print of a subset $S \subset \mathbb{R}^n \times [0, T]$, denoted $\operatorname{print}_r(S)$, is the set of points $(\gamma, \beta) \in (0, \infty]^2$ such that the quantity*

$$I_{\gamma,\beta}(S) := \left(\int_0^T \left(\int_{\{x | r_S(x,t) < r_0\}} r_S(x, t)^{-\gamma}\, \mathrm{d}x \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right)^{\frac{1}{\beta}} < \infty, \qquad (14)$$

where r_0 is some positive constant, $r_S(x,t) := \operatorname{dist}((x,t),S)$, and the appropriate integrals are interpreted as essential suprema if either $\gamma = \infty$ or $\beta = \infty$.

Observe that we integrate over the set

$$S_{r_0} := \{(x,t) \in \mathbb{R}^n \times [0,T] \,|\, r_S(x,t) < r_0\},$$

which is the r_0-neighbourhood of $S \subset \mathbb{R}^n \times [0,T]$. Further, note that the choice of positive constant r_0 is arbitrary as r_S^{-1} is bounded outside each neighbourhood of S. Equivalently, $(\gamma, \beta) \in \operatorname{print}_r(S)$ if and only if the quantity

$$\left\| r_S^{-1} \mathbf{1}_{S_{r_0}} \right\|_{L^\beta(0,T;L^\gamma(\mathbb{R}^n))} < \infty \tag{15}$$

where $\mathbf{1}_{S_{r_0}}$ is the characteristic function for the set S_{r_0}, but recall that (15) is not a norm for $\gamma, \beta < 1$.

The function r_S^{-1} is unbounded on S_{r_0} so for fixed γ, β the quantity $I_{\gamma,\beta}(S)$ is finite if r_S^{-1} is not 'too' singular. By allowing γ, β to vary we can capture a sense of how singular r_S^{-1} is, which quantifies the degree to which the set S is 'spread out'. Further, by allowing γ, β to vary independently we can weight the integrals so that the contribution from the spatial component is more or less significant than the contribution from the temporal component. Consequently, the r-codimension print encodes the degree to which the set S is spread out and the extent to which this spread is temporal rather than spatial.

This definition is easily generalised to consider the anisotropic detail of a subset $S \subset \mathbb{R}^{n+1}$ with respect to each of the $n+1$ coordinates: in this case the r-codimension print of S is the set of points $\alpha \in (0,\infty]^{n+1}$ such that the quantity

$$\left\| r_S^{-1} \mathbf{1}_{S_{r_0}} \right\|_{L^{\alpha_{n+1}}(\mathbb{R};L^{\alpha_n}(\mathbb{R};\ldots;L^{\alpha_1}(\mathbb{R})))} < \infty. \tag{16}$$

This broader definition more closely mimics the dimension prints of Rogers (1988) and Lee and Baek (1995) but for our application to non-autonomous ODEs we only wish to distinguish between the spatial and temporal detail of a subset. The 'spatio-temporal' r-codimension print of Definition 1.8 is simply the restriction of the more general r-codimension print to points $\alpha \in (0,\infty]^{n+1}$ such that $\alpha_1 = \alpha_2 = \ldots = \alpha_n$ so that the spatial contributions to the integral are all weighted equally.

Note that the definition of the r-codimension print presupposes an ordering of the coordinate axes in the order of integration of (16). It is not immediately clear how the print varies under reordering of axes. However, for our application we use the canonical spatio-temporal order presupposed in our choice of vector fields: the norm of the vector field $f \in L^p(0, T; L^q(\mathbb{R}^n))$ is defined by first integrating with respect to the spatial variables, then with respect to time. Henceforth, we consider $S \subset \mathbb{R}^n \times [0, T]$ and use Definition 1.8.

In Section 3 we establish some of the structure of the r-codimension print, including a relationship with the box-counting dimensions of S and its projections.

1.6. Motivation and applications

We briefly review two applications of the avoidance property for ordinary differential equations, the first of which is due to Nelson: the avoidance property was first studied in Nelson (1962) in order to establish the existence and uniqueness of generalised flow solutions for a narrow class of irregular vector fields. Below we discuss Nelson's results, his broader conjecture for solutions of irregular ODEs, and the limitations of this theory described in Aizenman (1978a).

For our second application we provide a condition for an irregular ODE to have a unique trajectory for almost all initial data. This application provides a useful supplement to the general theory of DiPerna & Lions (Theorem 1.3, above) as it is not currently possible to determine this stronger uniqueness property using their functional analytic approach to irregular ODEs.

1.6.1. Existence and uniqueness of generalised flows

In Nelson (1962) the author proposed a theory of irregular ODEs before the general theory of DiPerna & Lions was developed. Nelson examined autonomous ODEs and required a flow solution to consist of a trajectory for all initial data and satisfy (7) for all $x \in \mathbb{R}^n$, which is a stronger notion of solution than that used in the subsequent theory of DiPerna & Lions. In Nelson (1962) the author made the following conjecture:

Conjecture 1.9 (Nelson)**.** *If the vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has compact support, has zero (distributional) divergence, and $f \in L^2(\mathbb{R}^n)$ then there exists a unique map $X\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

- *for all $x \in \mathbb{R}^n$ the map $t \mapsto X(t, x)$ is a trajectory of* (1) *with initial data $x \in \mathbb{R}^n$, and*

- *X satisfies the group property* (7).

This conjecture was motivated by the success of energy methods for families of partial differential equations that are related to the transport equation $\frac{\partial u}{\partial t} = f \cdot \nabla u$ (which, for sufficiently regular vector fields, is equivalent to the ODE (1)). Nelson further suggests that the requirement $f \in L^2(\mathbb{R}^n)$ is significant by considering the following example: the radial vector field $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x) = -\frac{\operatorname{sign}(x_2)}{|x|^2} x \tag{17}$$

is divergenceless and satisfies $f \in L^p(\mathbb{R}^2)$ for p in the range $1 \leq p < 2$. However every trajectory of $\dot{x} = f(x)$ reaches the origin in finite time, so no collection of trajectories satisfies the group property (7) and consequently there is no generalised flow solution.

In the same paper the author proved a weakened form of the conjecture:

Theorem 1.10 (Nelson)**.** *There exists a unique generalised flow solution of the autonomous ODE $\dot{x} = f(x)$ if*

1) *f has compact support,*

2) *$f \in L^2(\mathbb{R}^n)$,*

3) *the distributional divergence* $\operatorname{div} f = 0$*, and*

4) *f is locally Lipschitz outside a closed set K of zero capacity.*

The proof has two components: first, as f is locally Lipschitz on $\mathbb{R}^n \setminus K$, there is a unique 'local' flow solution where each trajectory is defined on the largest time interval that the trajectory remains in $\mathbb{R}^n \setminus K$. Further, the condition (3) ensure that this local flow is measure preserving. As a generalised flow only requires a trajectory for almost all initial data it is sufficient to demonstrate that almost every trajectory is defined on the entire temporal domain and hence remains in $\mathbb{R}^n \setminus K$ where existence and uniqueness is assured. For a set $K \subset \mathbb{R}^n$ of Lebesgue measure zero this is precisely the requirement that the local flow avoids K. Nelson completes the proof of Theorem 1.10 by giving a sufficient condition for avoidance: with the regularity assumptions (1) - (3) a local flow avoids every set of zero capacity so in particular, from (4), avoids the set K.

Nelson's Conjecture 1.9 was proved false in Aizenman (1978a) through two counterexamples which are relevant to our examination of the avoidance property: in both cases Aizenman constructs a divergenceless vector field $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ with support contained in the unit cube

$$C := \left\{x \in \mathbb{R}^3 | 0 \leq x_i \leq 1, i = 1, 2, 3\right\}$$

for which there is a unique local flow: all initial data $x \in C$ gives rise to a unique trajectory defined for a short time, and further the aggregate of these trajectories is measure preserving. However, there is no generalised flow solution as, like Nelson's radial example (17), the distinct trajectories intersect in finite time, which violates the group property (7). In more detail, there exists a time $T > 0$ and a map

$$\gamma\colon\ [0, 1] \to \{x \in C | x_3 = 0\}$$

(that is taking values in the lower face of the cube) such that all initial data in the line segment $\{x \in C | x_2 = c, x_3 = 1\}$ gives rise to a distinct (unique) trajectory that intersects $\gamma(c)$ at time T.

The image of γ is precisely the image of the upper face $\{x \in C | x_3 = 1\}$ transported along the trajectories at time T. In terms of avoidance we see that, by construction, the local flow does not avoid the set $\operatorname{Im}(\gamma)$. Further it is precisely the set $\operatorname{Im}(\gamma)$ where the obstruction to the existence of a generalised flow is introduced as this is where the trajectories intersect.

Aizenman's first example has the above features and the further properties that

- the vector field f is bounded, and
- the map $\gamma\colon\ [0, 1] \to \{x \in C | x_3 = 0\}$ is surjective.

In this first example it is perhaps unsurprising that the local flow does not avoid $\operatorname{Im}(\gamma)$ as the set is a square transverse to the direction of the flow.

In his second example Aizenman investigates how small the set $\operatorname{Im}(\gamma)$ can be while retaining this non-existence result. He remarks that in order for a flow to be measure preserving as it approaching a small set $\operatorname{Im}(\gamma)$ the speed of the flow must increase. In particular, if $\mathcal{L}^2(\operatorname{Im}(\gamma)) = 0$ then the velocity field f must be singular at $\operatorname{Im}(\gamma)$. His second example takes parameters $m, k \in \mathbb{N}$ with $k \leq m^2$, has the above features and the further properties that

- the vector field $f \in L^p(\mathbb{R}^3)$ if and only if $p < \frac{\log m}{2\log m - \log k} + 1$, and
- the set $\mathrm{Im}(\gamma)$ has upper box-counting dimension

$$\dim_B(\mathrm{Im}(\gamma)) = \log k / \log m.$$

Aizenman highlights that this two-parameter family of examples provide borderline cases for the avoidance result of Theorem 1.5 as, for the set $\mathrm{Im}(\gamma)$, the avoidance criterion (9) holds for all $p > \frac{\log m}{2\log m - \log k} + 1$.

1.6.2. Almost everywhere uniqueness of trajectories

In Theorem 1.3, we recalled the key existence and uniqueness result for irregular ODEs from the seminal paper DiPerna and Lions (1989). This result is remarkable due to the generality of the hypotheses, its novel functional analytic approach and its subsequent extensions in Ambrosio (2004) and Crippa and De Lellis (2008).

We remark that Theorem 1.3 does not guarantee the stronger claim that almost every trajectory of the ODE (1) is unique. Indeed, there may be multiple trajectories of (1) for all initial data but a unique aggregate of these trajectories (up to equality almost everywhere) that satisfies the condition (4) and so forms a generalised flow solution. We illustrate this with a simple example in Robinson et al. (2012). Conversely, if almost every trajectory is unique then there is clearly a unique aggregate of these trajectories and so a unique generalised flow if this aggregate satisfies (4).

It is currently unknown if the functional analytic methods of DiPerna & Lions are able to determine almost everywhere uniqueness of trajectories. Ambrosio remarks that this problem is open even for autonomous vector fields with Sobolev regularity, and may be sensitive to the choice of representative in the equivalence class of f (Ambrosio (2004) pp.231). In fact, as we illustrate in Robinson et al. (2012), it is trivial to introduce non-uniqueness of trajectories by choosing a different representative of f.

In the following theorem we return to a more geometric viewpoint for irregular ODEs to provide a condition that guarantees the almost everywhere uniqueness of trajectories.

Theorem 1.11. *Let X be a generalised flow solution of the ODE* (1) *and let $S \subset \mathbb{R}^n \times [0,T]$ be a compact subset such that the trajectories of the ODE* (1) *are unique on sufficiently small time intervals for initial data $(x,s) \notin S$. If the flow X avoids the set S then for almost every $x \in \mathbb{R}^n$ the map $t \mapsto X(t,x,0)$ is the unique trajectory of* (1) *with initial data $(x,0)$.*

Proof. First we demonstrate that if a trajectory is not unique then it must intersect the set S. For a fixed $x \in \mathbb{R}^n$ suppose that there are two trajectories ξ_1, ξ_2 to the ODE (1) with initial data $(x, 0)$ and let

$$\tau := \sup \{t \in [0, T] \,|\xi_1 (u) = \xi_2 (u) \quad \forall u \leq t\}$$

be the latest time that the trajectories are identical. As the trajectories are distinct it is clear that $\tau \in (0, T)$ and further that ξ_1 and ξ_2 are two trajectories of (1) with initial data $(\xi_1 (\tau), \tau)$. Further, as these trajectories are distinct on each time interval $[\tau, \tau + \varepsilon)$ we conclude that $(\xi_1 (\tau), \tau) \in S$.

Consequently, if a trajectory does not intersect the set S then the trajectory must be unique. We conclude that if the generalised flow X avoids the set S then for almost every $x \in \mathbb{R}^n$ the trajectory $t \mapsto X(t, x, 0)$ is unique. □

In particular, we see from the Cauchy-Lipschitz Theorem that almost every trajectory is unique if a generalised flow solution avoids the set of points where the vector field f is not locally Lipschitz. This technique was recently used in Robinson and Sadowski (2009) (see Robinson et al. (2012) for a summary of the main ideas) to demonstrate that if f is a suitable weak solution of the Navier-Stokes equations then almost every trajectory is unique, and further that almost every trajectory is continuously differentiable. This result is physically significant as the trajectories describe the evolution of fluid particles.

2. Non-autonomous avoidance

We consider avoidance in the general non-autonomous case for an arbitrary bounded set $S \subset \mathbb{R}^n \times [0, T]$. We assume that the vector field $f \in L^p (0, T; L^q (\mathbb{R}^n))$ for some $p, q \in [1, \infty]$ but we make no further assumptions on the regularity of f and in particular we do not assume that (1)-(3) hold. Consequently, the vector field is not known to be sufficiently regular to guarantee the existence or uniqueness of a flow solution using the results of DiPerna and Lions (1989), and further we do not have the additional regularity $f \in L^1 \left(0, T; L^{n/(n-1)}_{\text{loc}} (\mathbb{R}^n)\right)$ that follows from (3). Instead, we assume that a generalised flow solution of (1) exists but we do not require this solution to be unique.

In the main result of this section, Theorem 2.3, we give an avoidance criterion in terms of the regularity of f and the r-codimension print of the set S. First, we give some trivial conditions for avoidance and non-avoidance in the following lemmas, where we introduce the notation

$$S^{\tau} := \{x \in \mathbb{R}^n | \, (x, \tau) \in S\}$$

for the temporal sections of a subset $S \subset \mathbb{R}^n \times [0, T]$.

Lemma 2.1. *If S has only a countable number of non-empty temporal sections, and every temporal section has zero n-dimensional Lebesgue measure then every generalised flow avoids S.*

Proof. Suppose $S = \bigcup_{i=1}^{\infty} S^{\tau_i} \times \{\tau_i\}$ with $\tau_i \in [0, T]$ and $\mathcal{L}^n \left(S^{\tau_i}\right) = 0$ for all $i \in \mathbb{N}$ and let X be a generalised flow. First, from (4), there exists a constant $C \in \mathbb{R}$ such that

$$\mathcal{L}^n \left(X \left(\tau_i, \cdot, 0\right)^{-1} S^{\tau_i}\right) \leq \mathrm{e}^{TC} \mathcal{L}^n \left(S^{\tau_i}\right) = 0 \quad \text{for all } i \in \mathbb{N}. \tag{18}$$

Next, the set of initial conditions at time 0 which give rise to trajectories that intersect a point of S

$$\begin{aligned} &\{x \in \mathbb{R}^n | \left(X \left(t, x, 0\right), t\right) \in S \quad \text{for some } t \in [0, T]\} \\ =& \{x \in \mathbb{R}^n | \left(X \left(\tau_i, x, 0\right), \tau_i\right) \in S \quad \text{for some } i \in \mathbb{N}\} \\ =& \bigcup_{i=1}^{\infty} \{x \in \mathbb{R}^n | X \left(\tau_i, x, 0\right) \in S^{\tau_i}\} = \bigcup_{i=1}^{\infty} X \left(\tau_i, \cdot, 0\right)^{-1} S^{\tau_i}, \end{aligned}$$

which from (18) is the countable union of null sets and so has zero n-dimensional Lebesgue measure, so the flow X avoids the set S. $\square$

Conversely, if a temporal section of S has positive measure then no generalised flow avoids S:

Lemma 2.2. *If $S \subset \mathbb{R}^n \times [0, T]$ has a temporal section S^{τ} of positive n-dimensional measure for some $\tau \in [0, T]$ then no generalised flow avoids S.*

Proof. Suppose $\mathcal{L}^n \left(S^{\tau}\right) > 0$ and let X be a generalised flow. First, from (4) there exists a constant $C \in \mathbb{R}$ such that

$$\mathcal{L}^n \left(X \left(\tau, \cdot, 0\right)^{-1} S^{\tau}\right) \geq \mathrm{e}^{-TC} \mathcal{L}^n \left(S^{\tau}\right) > 0. \tag{19}$$

Next, the set

$$\{x \in \mathbb{R}^n |\, (X\,(t, x, 0)\,, t) \in S \quad \text{for some } t \in [0, T]\}$$

contains

$$\begin{aligned}\{x \in \mathbb{R}^n |\, (X\,(\tau, x, 0)\,, \tau) \in S\} &= \{x \in \mathbb{R}^n | X\,(\tau, x, 0) \in S^\tau\} \\ &= X\,(\tau, \cdot, 0)^{-1}\, S^\tau.\end{aligned}$$

which, from (19) has positive n-dimensional Lebesgue measure. $\square$

In light of the above lemma no avoidance criterion can be satisfied by a set with a temporal section of positive measure. Fortunately, such sets have particular r-codimension prints: in Lemma 3.1 we demonstrate that the print of these sets does not contain the point $(1, 1)$. This result simplifies the proof of the following theorem, in which we adapt the avoidance result of Aizenman (1978b) to the non-autonomous case. As in Cipriano and Cruzeiro (2005) our proof follows the reasoning of Aizenman (1978b). The proof requires some further technical results in the geometry of the r-codimension print, which we delay until Section 3.

Theorem 2.3. *Let $X\colon\ [0, T] \times \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ be a generalised flow solution of the ODE $\dot{x} = f\,(x, t)$ where $f \in L^p\,(0, T; L^q\,(\mathbb{R}^n))$ for $1 \le p, q \le \infty$ and let S be a compact subset of $\mathbb{R}^n \times [0, T]$. If the pair of Hölder conjugates (q^*, p^*) is in the r-codimension print of S then the flow X avoids the subset S.*

Proof. As the pair of Hölder conjugates (q^*, p^*) is in $\text{print}_r\,(S)$ and $p^*, q^* \ge 1$ then, from property (viii) of Lemma 3.1, the point $(1, 1)$ is also in $\text{print}_r\,(S)$. Consequently, from property (iii) of Lemma 3.1 the set S does not have a temporal section with positive n-dimensional Lebesgue measure, and in particular, $\mathcal{L}^n\,(S^0) = 0$.

For brevity, denote

$$\Omega := \{x \in \mathbb{R}^n |\, (X\,(t, x, 0)\,, t) \in S \quad \text{for some } t \in [0, T]\} \setminus S^0$$

the set of spatial points at time 0 that do not lie in the time 0 temporal slice of S but which give rise to trajectories that intersect S. We denote by Ω^{cont} the set of $x \in \Omega$ that give rise to absolutely continuous trajectories. As $\mathcal{L}^n\,(S^0) = 0$ and, from the definition of the flow, $\mathcal{L}^n\,(\Omega \setminus \Omega^{\text{cont}}) = 0$ it is sufficient to demonstrate that Ω^{cont} has zero Lebesgue measure.

Following Aizenman (1978b), for $\delta > 0$ and $x \in \Omega^{\text{cont}}$ we define

$$\tau_\delta(x) := \begin{cases} \sup\{u | r_S(X(t,x,0),t) \geq \delta \quad \forall t \in [0,u]\} & r_S(x,0) > \delta \\ 0 & r_S(x,0) \leq \delta \end{cases}$$

the latest time for which the trajectory from x is outside the δ-neighbourhood of S. Clearly $\tau_\delta(x) \leq T$ for all $x \in \Omega^{\text{cont}}$ as the continuous trajectories intersect S by time T. Further, from the continuity of the trajectories it is clear that

$$r_S(X(\tau_\delta(x),x,0),\tau_\delta(x)) = \delta \qquad \forall x \in \Omega^{\text{cont}} \quad \text{with} \quad r_S(x,0) > \delta. \tag{20}$$

Finally, note that $r_S(x,0) > 0$ for all $x \in \Omega^{\text{cont}}$ as $r_S(x,0) = 0$ only if $x \in S^0$. Consequently, for all $\delta > 0$ the set Ω^{cont} is contained in a countable union of sets of the form

$$\Omega_{m,\delta} := \left\{x \in \Omega^{\text{cont}} | r_S(x,0) \geq 1/m, \quad \tau_\delta(x) \leq T\right\}.$$

Fix $r_0 > 0$ and $0 < \delta < r_0$ and let

$$F(\delta) := \left\{x \in \Omega^{\text{cont}} | r_S(x,0) \geq r_0, \quad \tau_\delta(x) < T\right\}.$$

We now show that $\mathcal{L}^n(F(\delta)) \to 0$ as $\delta \to 0$: first, introduce the Lipschitz function

$$g(y) = \begin{cases} \log\left(\frac{r_0}{y}\right) & \delta \leq y \leq r_0 \\ 0 & r_0 < y \end{cases}$$

chosen so that $g(r_S(x,0)) = 0$ for $x \in F(\delta)$, and from (20),

$$g(r_S(X(\tau_\delta(x),x,0),\tau_\delta(x))) = g(\delta)$$

so that

$$\mathcal{L}^n(F(\delta))|g(\delta)| = \int_{F(\delta)} |g(r_S(X(\tau_\delta(x),x,0),\tau_\delta(x))) - g(r_S(x,0))| \,\mathrm{d}x. \tag{21}$$

Next, as

$$|r_S(X(t_1,x,0),t_1) - r_S(X(t_2,x,0),t_2)| \leq |X(t_1,x,0) - X(t_2,x,0)| + |t_1 - t_2|$$

the map $t \mapsto r_S(X(t,x,0),t)$ is absolutely continuous and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t} r_S(X(t,x,0),t)\right| \leq \left|\frac{\mathrm{d}}{\mathrm{d}t} X(t,x,0)\right| + 1.$$

Consequently, the composition $g(r_S(X(t,x,0),t))$ is absolutely continuous in t and so, from the chain rule for almost everywhere differentiable functions (see Serrin and Varberg (1969)), for almost every t

$$\frac{\mathrm{d}}{\mathrm{d}t} g(r_S(X(t,x,0),t)) = g'(r_S(X(t,x,0),t)) \frac{\mathrm{d}}{\mathrm{d}t} r_S(X(t,x,0),t).$$

From (21) we write

$$\begin{aligned}
\mathcal{L}^n(F(\delta))\,|g(\delta)| &= \int_{F(\delta)} \left| \int_0^{\tau_\delta(x)} \frac{\mathrm{d}}{\mathrm{d}t} g(r_S(X(t,x,0),t))\,\mathrm{d}t \right| \mathrm{d}x \\
&\leq \int\limits_{F(\delta)} \int_\rho^{\tau_\delta(x)} |g'(r_S(X(t,x,0),t))| \left(\left| \frac{\mathrm{d}}{\mathrm{d}t} X(t,x,0) \right| + 1 \right) \mathrm{d}t\,\mathrm{d}x \\
&= \int\limits_{F(\delta)} \int_0^{\tau_\delta(x)} |g'(r_S(X(t,x,0),t))| \left(|f(X(t,x,0),t)| + 1 \right) \mathrm{d}t\,\mathrm{d}x \\
&\leq \mathrm{e}^{TC} \int\limits_{F(\delta)} \int_0^{\tau_\delta(x)} |g'(r_S(x,t))|\,|f(x,t)+1|\,\mathrm{d}t\,\mathrm{d}x
\end{aligned}$$

where we use the fact that X is a generalised flow satisfying (4) and $C \in \mathbb{R}$ is the constant from (4), from which Fubini's Theorem yields

$$\mathcal{L}^n(F(\delta))\,|g(\delta)| \leq \mathrm{e}^{TC} \int_0^T \int\limits_{F(\delta)} |g'(r(x,t))|\,|f(x,t)+1|\,\mathrm{d}x\,\mathrm{d}t. \tag{22}$$

Next, as $|g(\delta)| = \log\left(\frac{r_0}{\delta}\right)$ and the derivative

$$g'(y) = \begin{cases} -\frac{1}{y} & \text{for almost every } y \in (\delta, r_0] \\ 0 & \text{for almost every } y > r_0, \end{cases}$$

the inequality (22) is

$$\mathcal{L}^n(F(\delta)) \leq \mathrm{e}^{TC} \log\left(\frac{r_0}{\delta}\right)^{-1} \int_0^T \int\limits_{\{x | r_S(x,t) < r_0\}} r_S(x,t)^{-1}\,|f(x,t)+1|\,\mathrm{d}x\,\mathrm{d}t$$

which, after applying Hölder's inequality, gives

$$\mathcal{L}^n(F(\delta)) \le \mathrm{e}^{TC} \log\left(\frac{r_0}{\delta}\right)^{-1} I_{q^*,p^*}(S) \left\|(f+1)\mathbf{1}_{S_{r_0}}\right\|_{L^p(0,T;L^q(\mathbb{R}^n))}.$$

This is finite as $(q^*, p^*) \in \mathrm{print}_r(S)$ and $f+1 \in L^p(0,T;L^q_{loc}(\mathbb{R}^n))$, so $I_{q^*,p^*}(S)$ and $\left\|(f+1)\mathbf{1}_{S_{r_0}}\right\|_{L^p(0,T;L^q(\mathbb{R}^n))}$ are both finite. As $\delta > 0$ was arbitrary we let $\delta \to 0$ whence $\log\left(\frac{r_0}{\delta}\right)^{-1} \to 0$ giving the desired result. $\square$

3. Geometry of the r-codimension print

In the previous section we demonstrated that a generalised flow solution avoids a set $S \subset \mathbb{R}^n \times [0,T]$ if the r-codimension print of S contains a particular point. In light of this result we wish to be able to determine the r-codimension print of a given set. In this section we derive some elementary properties of the r-codimension print, provide inclusion and exclusion conditions in terms of the more familiar box-counting dimension, and provide some examples in which we compute the prints of some simple sets.

The elementary properties of the r-codimension print are described in the following lemma:

Lemma 3.1. *Let $S, S_i \subset \mathbb{R}^n \times [0,T]$ be bounded non-empty subsets for $i \in \mathbb{N}$,*

(i) $\mathrm{print}_r(S)$ *is a subset of the union*

$$\begin{aligned} &\{(\gamma,\beta)\,|\gamma\beta < \gamma + \beta n \quad 0 < \gamma, \beta < \infty\} \\ \cup &\{(\gamma,\infty)\,|0 < \gamma < n\} \\ \cup &\{(\infty,\beta)\,|0 < \beta < 1\} \end{aligned}$$

illustrated in Figure 1,

(ii) *if* $\mathcal{L}^{n+1}(S) > 0$ *then* $\mathrm{print}_r(S) = \emptyset$,

(iii) *if* $\mathcal{L}^n(S^\tau) > 0$ *for some* $\tau \in [0,T]$ *then*

$$\mathrm{print}_r(S) \subset \{(\gamma,\beta)\,|0 < \beta < 1\},$$

(iv) *if* $S_1 \subset S_2$ *then* $\mathrm{print}_r(S_2) \subset \mathrm{print}_r(S_1)$,

(v) $\mathrm{print}_r(\cup_{i=1}^\infty S_i) \subset \cap_{i=1}^\infty \mathrm{print}_r(S_i)$,

(vi) *S and its closure* $\mathrm{cl}(S)$ *have the same r-codimension print,*

(vii) *if $y \in \mathbb{R}^n$, $s \in [-T, T]$ and $\lambda > 0$ are such that the sets*

$$S + (y, s) := \{(x + y, t + s) \mid (x, t) \in S\}$$
$$\lambda S := \{(\lambda x, \lambda t) \mid (x, t) \in S\}$$

are subsets of $\mathbb{R}^n \times [0, T]$ then

$$\mathrm{print}_r(S) = \mathrm{print}_r(S + (y, s)) = \mathrm{print}_r(\lambda S),$$

and

(viii) *if $(\gamma, \beta) \in \mathrm{print}_r(S)$ then the rectangle $(0, \gamma] \times (0, \beta] \subset \mathrm{print}_r(S)$.*

Proof. The properties (iv), (vii) and (vi) follow from the observations that for $S_1 \subset S_2$

$$r_{S_1}(x, t) \geq r_{S_2}(x, t), \qquad r_S(x, t) = r_{\mathrm{cl}(S)}(x, t) \qquad \text{and}$$
$$r_S(x, t) = \frac{1}{\lambda} r_{\lambda S}(\lambda x, \lambda t) = r_{S+(y,s)}(x + y, t + s),$$

and an inductive application of (iv) yields (v).

The property (viii) immediately follows from Hölder's inequality.

From (iv) and (vii) it is clear that $\mathrm{print}_r(S) \subset \mathrm{print}_r(\{0\})$, which is equal to the union given in (i), although we delay this calculation until Example 3.2.

Next, if $\mathcal{L}^{n+1}(S) > 0$ then $r_S(x, t)^{-1}$ is unbounded on a set of positive $(n+1)$-dimensional Lebesgue measure in which case the integral (14) is infinite for all γ, β, yielding (ii).

Finally, suppose that $\mathcal{L}^n(S^\tau) > 0$ for some $\tau \in [0, T]$. Clearly $S^\tau \times \{\tau\} \subset S$ so from (iv)

$$\mathrm{print}_r(S^\tau \times \{\tau\}) \supset \mathrm{print}_r(S). \tag{23}$$

Further,

$$S^\tau \times [\tau - r_0, \tau + r_0] \subset (S^\tau \times \{\tau\})_{r_0}, \tag{24}$$

and

$$r_{S^\tau \times \{\tau\}}(x, t) = |t - \tau| \quad \text{for all } x \in S^\tau. \tag{25}$$

Consequently,

$$
\begin{aligned}
I_{\infty,\beta}\left(S^{\tau}\times\{\tau\}\right) &= \left(\int_0^T \left(\operatorname*{ess\,sup}_{\left\{x|r_{S^{\tau}\times\{\tau\}(x,t)<r_0}\right\}} r_{S^{\tau}\times\{\tau\}}\left(x,t\right)^{-1}\right)^{\beta} \mathrm{d}t\right)^{\frac{1}{\beta}} \\
&\geq \left(\int_{\tau-r_0}^{\tau+r_0} \left(\operatorname*{ess\,sup}_{x\in S^{\tau}} r_{S^{\tau}\times\{\tau\}}\left(x,t\right)\right)^{\beta} \mathrm{d}t\right)^{\frac{1}{\beta}}
\end{aligned}
$$

from (24), which from (25),

$$
= \left(\int_{\tau-r_0}^{\tau+r_0} \left|t-\tau\right|^{-\beta} \mathrm{d}t\right)^{\frac{1}{\beta}},
$$

which diverges for all $\beta \geq 1$. Consequently, from (viii) no point (γ,β) with $\beta \geq 1$ is in $\mathrm{print}_r\left(S^{\tau}\times\{\tau\}\right)$. We conclude from (23) that

$$
\mathrm{print}_r\left(S\right) \subset \mathrm{print}_r\left(S^{\tau}\times\{\tau\}\right) \subset \left\{(\gamma,\beta)\left|0<\beta<1\right.\right\},
$$

yielding (iii). □

Our use of the term 'codimension' is justified by the reversal of inclusions (iv), which is a property shared by the more familiar codimensions $n-\dim\left(A\right)$.

The following lemma provides necessary and sufficient conditions for the integral

$$
\int_{A_{r_0}} r_A\left(x\right)^{-\gamma} \mathrm{d}x
$$

to be finite for a bounded set $A \subset \mathbb{R}^n$. This allows us to include some points and exclude others from the r-codimension print of a set $S \subset \mathbb{R}^n \times [0,T]$, which is the content of Corollary 3.3. The sufficient condition, due to Aizenman (1978b), is in terms of the upper box-counting dimension of A, while the necessary condition is in terms of the lower box-counting dimension.

Lemma 3.2. *For a bounded set $A \subset \mathbb{R}^n$ and any $r_0 > 0$ the integral*

$$
\int_{A_{r_0}} r_A\left(x\right)^{-\gamma} \mathrm{d}x \tag{26}
$$

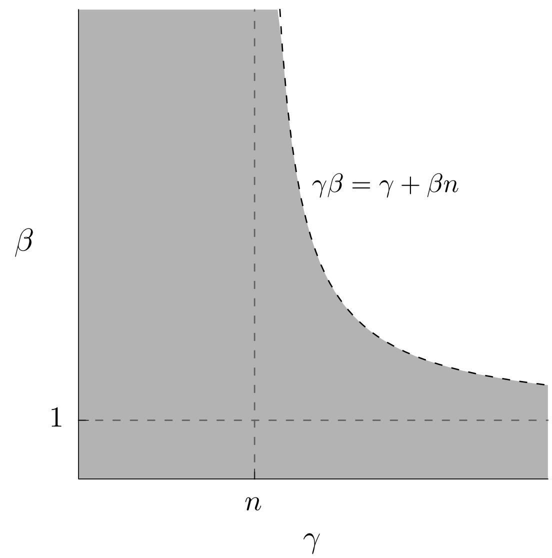

Figure 1: The r-codimension print of the singleton $\{0\} \subset \mathbb{R}^n \times [0, T]$.

- *is finite if* $0 \leq \gamma < n - \dim_B(A)$, *and*
- *is infinite if* $n - \dim_{LB}(A) < \gamma$.

Proof. We split the integral into the minimum value for $r_A(x)^{-\gamma}$ and the difference between the minimum and actual value

$$\int_{A_{r_0}} r_A(x)^{-\gamma}\,\mathrm{d}x = \int_{A_{r_0}} r_0^{-\gamma}\mathrm{d}x + \int_{A_{r_0}} \left[r_A(x)^{-\gamma} - r_0^{-\gamma}\right]\mathrm{d}x$$

The second integral we rewrite as

$$\int_{A_{r_0}} \left[r_A(x)^{-\gamma} - r_0^{-\gamma}\right]\mathrm{d}x = \int_{A_{r_0}} \int_{r_0^{-\gamma}}^{r_A(x)^{-\gamma}} 1\,\mathrm{d}u\,\mathrm{d}x$$

which, from Fubini's Theorem,

$$= \int_{r_0^{-\gamma}}^{\infty} \int_{\left\{x \middle| r_A(x) < u^{-\frac{1}{\gamma}}\right\}} 1\,\mathrm{d}x\,\mathrm{d}u$$

$$= \int_{r_0^{-\gamma}}^{\infty} \mathcal{L}^n\left(A_{u^{-\frac{1}{\gamma}}}\right)\mathrm{d}u,$$

so we rewrite the integral (26) as

$$\int_{A_{r_0}} r_A(x)^{-\gamma}\,\mathrm{d}x = r_0^{-\gamma}\mathcal{L}^n(A_{r_0}) + \int_{r_0^{-\gamma}}^{\infty} \mathcal{L}^n\left(A_{u^{-\frac{1}{\gamma}}}\right)\mathrm{d}u. \tag{27}$$

First, we assume that $0 \leq \gamma < n - \dim_B(A)$ and let $\varepsilon > 0$ be sufficiently small that $\gamma + \varepsilon < n - \dim_B(A)$. From Lemma 1.6 there exists a constant $C > 0$ such that $\mathcal{L}^n\left(A_{u^{-\frac{1}{\gamma}}}\right) \leq C\left(u^{-\frac{1}{\gamma}}\right)^{\gamma+\varepsilon}$ for all $u^{-\frac{1}{\gamma}} < r_0$. Consequently, the integral (26) is bounded above

$$\int_{A_{r_0}} r_A(x)^{-\gamma}\,\mathrm{d}x \leq r_0^{-\gamma}\mathcal{L}^n(A_{r_0}) + \int_{r_0^{-\gamma}}^{\infty} Cu^{-\left(1+\frac{\varepsilon}{\gamma}\right)}\mathrm{d}u,$$

which is finite as $1 + \frac{\varepsilon}{\gamma} > 1$.

Next, we assume that $n - \dim_{LB}(A) < \gamma$. Again from Lemma 1.6 there exists a constant $C > 0$ such that $\mathcal{L}^n\left(A_{u^{-\frac{1}{\gamma}}}\right) \geq \frac{1}{C}\left(u^{-\frac{1}{\gamma}}\right)^{\gamma}$ for all $u^{-\frac{1}{\gamma}} < r_0$. Consequently the integral (26) is bounded below

$$\int_{A_{r_0}} r_A(x)^{-\gamma}\,\mathrm{d}x \geq r_0^{-\gamma}\mathcal{L}^n(A_{r_0}) + \int_{r_0^{-\gamma}}^{\infty} \frac{1}{C}u^{-1}\mathrm{d}u,$$

which is infinite as the final integral diverges. □

Corollary 3.3. *For a bounded subset $S \subset \mathbb{R}^n \times [0, T]$ every point of the open square $(0, n+1-\dim_B(S))^2$ is in r-codimension print of S. Further, every point of the square $(n+1-\dim_{LB}(S), \infty]^2$ is not in the r-codimension print of S. These points are illustrated in Figure 2.*

Proof. Follows from the previous corollary and property (viii) of Lemma 3.1. □

It is immediate from Figure 2 that there is a gap between the inclusion and exclusion criteria of Corollary 3.3: indeed, we are unable to determine from the corollary if the point (γ, γ) is in $\mathrm{print}_r(S)$ for γ in the range

$$n + 1 - \dim_B(S) \leq \gamma \leq n + 1 - \dim_{LB}(S).$$

As there are sets for which $\dim_{LB}(S) = 0$ and $\dim_B(S) = n$ (see Robinson and Sharples (2012)), this gap can be large. In the following section we supplement the inclusion and exclusion criteria of Corollary 3.3 by considering the box-counting dimensions of the projections of S.

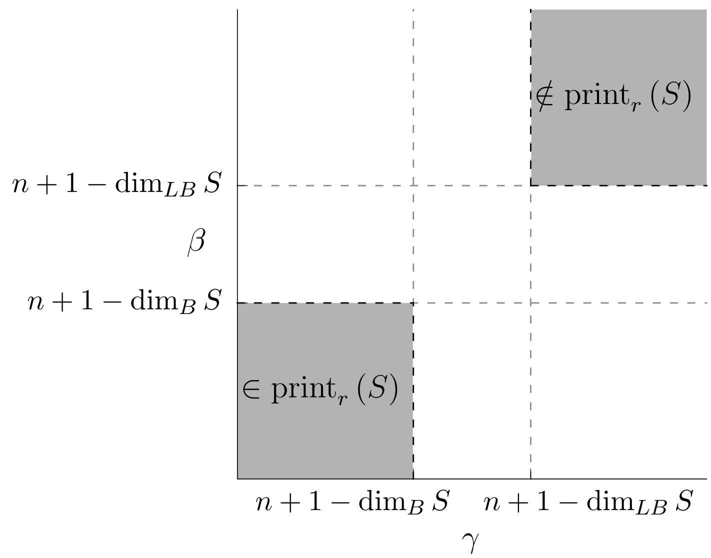

Figure 2: A subset of points (γ, β) that are in $\mathrm{print}_r(S)$ and a subset of points (γ, β) that are not in $\mathrm{print}_r(S)$.

3.1. Product sets

We now consider sets of the form $S := A \times \mathcal{T}$ where $A \subset \mathbb{R}^n$ is bounded and $\mathcal{T} \subset [0, T]$. With this product structure we can write the distance $r_S(x,t)$ in terms of the distance from x to A and the distance from t to $\mathcal{T}$: we introduce the notation $r_A(x)$ and $r_\mathcal{T}(t)$ for these respective distances and note that

$$r_S(x,t)^2 = r_A(x)^2 + r_\mathcal{T}(t)^2 . \tag{28}$$

In the following theorem we provide conditions for points to be in the r-codimension print of a product set. Conditions (i) and (ii) are consequences of Lemma 3.2; our interest is in conditions (iii) and (iv).

Theorem 3.4. *Let $A \subset \mathbb{R}^n$ be bounded, $\mathcal{T} \subset [0, T]$ and let $S := A \times \mathcal{T}$. The point (γ, β) is in $\mathrm{print}_r(S)$ if one of the following conditions holds:*

(i) $\gamma < n - \dim_B(A)$

(ii) $\beta < 1 - \dim_B(\mathcal{T})$

(iii) $\gamma\beta < \gamma(1 - \dim_B(\mathcal{T})) + \beta(n - \dim_B(A))$.

Further, the point (γ, β) *is not in* $\mathrm{print}_r(S)$ *if the following condition holds*

(iv) $\gamma\beta > \gamma\left(1 - \dim_{LB}(\mathcal{T})\right) + \beta\left(n - \dim_{LB}(A)\right)$.

These points are represented in Figure 3.

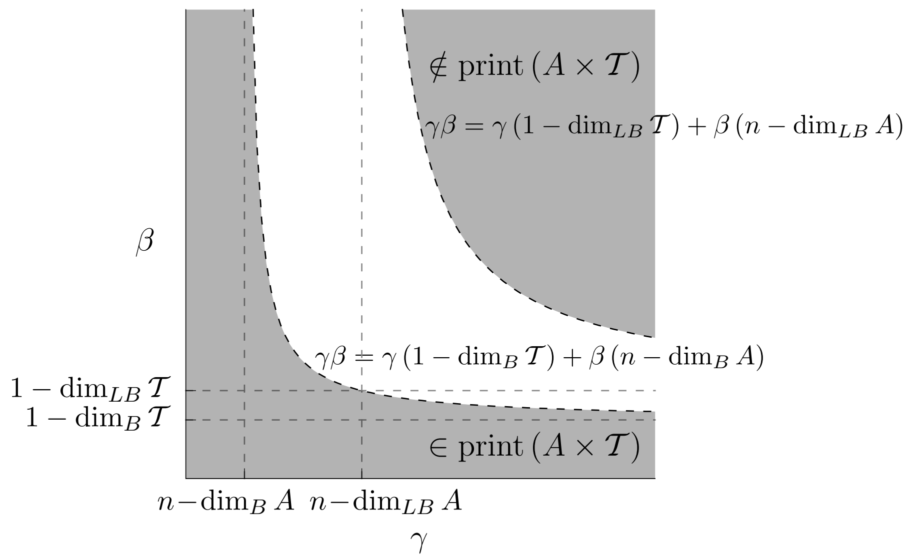

Figure 3: The result of Theorem 3.4: the region below the lower hyperbola consists of points $(\gamma, \beta) \in \mathrm{print}_r(A \times \mathcal{T})$; the region above the upper hyperbola consists of points $(\gamma, \beta) \notin \mathrm{print}_r(A \times \mathcal{T})$. The theorem provides no information about points on the hyperbolas themselves or in the region between them.

Proof. Note that in light of the equality (28)

$$I_{\gamma,\beta}(S) = \left(\int_0^T \left(\int_{\left\{x \middle| r_A(x)^2 + r_{\mathcal{T}}(t)^2 < r_0^2\right\}} \left(r_A(x)^2 + r_{\mathcal{T}}(t)^2\right)^{-\frac{\gamma}{2}} \mathrm{d}x \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right)^{\frac{1}{\beta}}, \quad (29)$$

where again the appropriate integrals are interpreted as essential suprema if $\gamma = \infty$ or $\beta = \infty$.

First we assume that condition (i) holds. Consider

$$\begin{aligned}
I_{\gamma,\infty}(S) &= \operatorname*{ess\,sup}_{t\in[0,T]} \left(\int\limits_{\left\{x \mid r_A(x)^2 + r_{\mathcal{T}}(t)^2 < r_0^2\right\}} \left(r_A(x)^2 + r_{\mathcal{T}}(t)^2\right)^{-\frac{\gamma}{2}} \mathrm{d}x \right)^{\frac{1}{\gamma}} \\
&\leq \operatorname*{ess\,sup}_{t\in[0,T]} \left(\int\limits_{\{x \mid r_A(x) < r_0\}} r_A(x)^{-\gamma} \mathrm{d}x \right)^{\frac{1}{\gamma}} \\
&= \left(\int_{A_{r_0}} r_A(x)^{-\gamma} \mathrm{d}x \right)^{\frac{1}{\gamma}} < \infty
\end{aligned}$$

from Lemma 3.2 as $\gamma < n - \dim_B(A)$. Consequently, from property (viii) of Lemma 3.1, $(\gamma, \beta) \in \operatorname{print}_r(S)$ for all $\beta \in (0, \infty]$.

Next we assume that condition (ii) holds. Consider

$$\begin{aligned}
I_{\infty,\beta}(S) &= \left(\int_0^T \left(\operatorname*{ess\,sup}_{\left\{x \mid r_A(x)^2 + r_{\mathcal{T}}(t)^2 < r_0\right\}} \left(r_A(x)^2 + r_{\mathcal{T}}(t)^2\right)^{-\frac{1}{2}} \right)^{\beta} \mathrm{d}t \right)^{\frac{1}{\beta}} \\
&\leq \left(\int_0^T \left(\operatorname*{ess\,sup}_{\{x \mid r_{\mathcal{T}}(t) < r_0\}} r_{\mathcal{T}}(t)^{-1} \right)^{\beta} \mathrm{d}t \right)^{\frac{1}{\beta}} \\
&= \left(\int_{\mathcal{T}_{r_0}} r_{\mathcal{T}}(t)^{-\beta} \mathrm{d}t \right)^{\frac{1}{\beta}} < \infty
\end{aligned}$$

from Lemma 3.2, as $\beta < 1 - \dim_B(\mathcal{T})$. As above, it follows that $(\gamma, \beta) \in \operatorname{print}_r(S)$ for all $\gamma \in (0, \infty]$.

For conditions (iii) and (iv) both γ and β are finite. We write (29) as

$$I_{\gamma,\beta}(S) = \left(\int_{\mathcal{T}_{r_0}} \left(\int_{A_{\sqrt{r_0^2 - r_{\mathcal{T}}(t)^2}}} \left(r_X(x)^2 + r_{\mathcal{T}}(t)^2\right)^{-\frac{\gamma}{2}} \mathrm{d}x \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right)^{\frac{1}{\beta}}$$

and for each $t \in \mathcal{T}_{r_0}$ we define

$$J(t) := \int_{A_{\sqrt{r_0^2 - r_{\mathcal{T}}(t)^2}}} \left(r_A(x)^2 + r_{\mathcal{T}}(t)^2\right)^{-\frac{\gamma}{2}} \mathrm{d}x$$

so that

$$I_{\gamma,\beta}(S) = \left(\int_{\mathcal{T}_{r_0}} J(t)^{\frac{\beta}{\gamma}} \mathrm{d}t\right)^{\frac{1}{\beta}}. \tag{30}$$

Fix $t \in \mathcal{T}_{r_0}$ and, proceeding in a similar fashion to the proof of Lemma 3.2, we write $J(t)$ as the sum

$$J(t) = \int_{A_{\sqrt{r_0^2 - r_{\mathcal{T}}(t)^2}}} r_0^{-\gamma} \mathrm{d}x \tag{31}$$

$$+ \int_{A_{\sqrt{r_0^2 - r_{\mathcal{T}}(t)^2}}} \left(r_A(x)^2 + r_{\mathcal{T}}(t)^2\right)^{-\frac{\gamma}{2}} - r_0^{-\gamma} \mathrm{d}x. \tag{32}$$

The second integral (32) is equal to

$$\int_{A_{\sqrt{r_0^2 - r_{\mathcal{T}}(t)^2}}} \int_{r_0^{-\gamma}}^{\left(r_A(x)^2 + r_{\mathcal{T}}(t)^2\right)^{-\frac{\gamma}{2}}} 1 \mathrm{d}u \, \mathrm{d}x = \int_{r_0^{-\gamma}}^{r_{\mathcal{T}}(t)^{-\gamma}} \int_{A_{\sqrt{u^{-\frac{2}{\gamma}} - r_{\mathcal{T}}(t)^2}}} 1 \mathrm{d}x \, \mathrm{d}u$$

from Fubini's Theorem. Consequently,

$$J(t) = r_0^{-\gamma} \mathcal{L}^n\left(A_{\sqrt{r_0^2 - r_{\mathcal{T}}(t)^2}}\right) + \int_{r_0^{-\gamma}}^{r_{\mathcal{T}}(t)^{-\gamma}} \mathcal{L}^n\left(A_{\sqrt{u^{-\frac{2}{\gamma}} - r_{\mathcal{T}}(t)^2}}\right) \mathrm{d}u$$

so from (30)

$$I_{\gamma,\beta}(S) = \left(\int_{\mathcal{T}_{r_0}} \left(r_0^{-\gamma} \mathcal{L}^n\left(A_{\sqrt{r_0^2 - r_{\mathcal{T}}(t)^2}}\right) + \int_{r_0^{-\gamma}}^{r_{\mathcal{T}}(t)^{-\gamma}} \mathcal{L}^n\left(A_{\sqrt{u^{-\frac{2}{\gamma}} - r_{\mathcal{T}}(t)^2}}\right) \mathrm{d}u\right)^{\frac{\beta}{\alpha}} \mathrm{d}t\right)^{\frac{1}{\beta}}. \tag{33}$$

Next, we assume that condition (iii) holds. In light of the previous two cases, we assume additionally that $\gamma \geq n - \dim_B(A)$ and that $n - \dim_B(A) > 0$ as condition (iii) reduces to (ii) if $\dim_B(X) = n$. With these assumptions there exists an η such that $0 \leq \eta < n - \dim_B(A)$ and

$$\gamma\beta < \gamma\left(1 - \dim_B(\mathcal{T})\right) + \beta\eta. \tag{34}$$

Consequently, from Lemma 1.6 there exists a constant C such that $\mathcal{L}^n(A_\delta) \leq C\delta^\eta$ for all $0 < \delta \leq r_0$. From (33),

$$\begin{aligned}
I_{\gamma,\beta}(S) &\leq \int_0^T \left[r_0^{-\gamma} C \left(r_0^2 - r_{\mathcal{T}}(t)^2\right)^{\frac{\eta}{2}} + \int\limits_{r_0^{-\gamma}}^{r_{\mathcal{T}}(t)^{-\gamma}} C \left(u^{-\frac{2}{\gamma}} - r_{\mathcal{T}}(t)^2\right)^{\frac{\eta}{2}} \mathrm{d}u \right]^{\frac{\beta}{\gamma}} \mathrm{d}t \Bigg]^{\frac{1}{\beta}} \\
&\leq C^{\frac{1}{\gamma}} \left[\int_0^T \left(r_0^{\eta-\gamma} + \int\limits_{r_0^{-\gamma}}^{r_{\mathcal{T}}(t)^{-\gamma}} u^{-\frac{\eta}{\gamma}} \mathrm{d}u \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}}
\end{aligned}$$

and, as $\gamma > \eta$,

$$\begin{aligned}
&\leq C^{\frac{1}{\gamma}} \left[\int_0^T \left(r_0^{\eta-\gamma} + \frac{1}{1 - \frac{\eta}{\gamma}} \left(r_{\mathcal{T}}(t)^{\eta-\gamma} - r_0^{\eta-\gamma} \right) \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}} \\
&\leq \left(\frac{C}{1 - \frac{\eta}{\gamma}} \right)^{\frac{1}{\gamma}} \left[\int_0^T \left(r_{\mathcal{T}}(t)^{\eta-\gamma} \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}} \\
&= \left(\frac{C}{1 - \frac{\eta}{\gamma}} \right)^{\frac{1}{\gamma}} \left[\int_0^T r_{\mathcal{T}}(t)^{\frac{\beta(\eta-\gamma)}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}} < \infty
\end{aligned}$$

from Lemma 3.2 as it follows from (34) that $0 \leq (\beta\gamma - \beta\eta)/\gamma < 1 - \dim_B(\mathcal{T})$.

Next, assume that condition (iv) holds so there exists an η such that

$$\begin{aligned}
&\eta > n - \dim_{LB}(A) \\
\text{and} \qquad &\gamma\beta > \gamma\left(1 - \dim_{LB}(\mathcal{T})\right) + \beta\eta.
\end{aligned} \tag{35}$$

From Lemma 1.6 there exists a constant C such that $\mathcal{L}^n(A_\delta) \geq C^{-1}\delta^\eta$ for all $0 < \delta \leq r_0$ and consequently, from (33),

$$I_{\gamma,\beta}(S) \geq C^{-\frac{1}{\gamma}} \left[\int\limits_{\mathcal{T}_{r_0}} \left(\int\limits_{r_0^{-\gamma}}^{r_{\mathcal{T}}(t)^{-\gamma}} \left(u^{-\frac{2}{\gamma}} - r_{\mathcal{T}}(t)^2 \right)^{\frac{\eta}{2}} \mathrm{d}u \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}}.$$

By restricting the domain of the first integral to $\mathcal{T}_{r_0/\sqrt{2}}$ and the domain of the second to u such that $r_0^{-\gamma} \leq u \leq \left(\sqrt{2} r_{\mathcal{T}}(t)\right)^{-\gamma}$, we write

$$I_{\gamma,\beta}(S) \geq C^{-\frac{1}{\gamma}} \left[\int\limits_{\mathcal{T}_{r_0/\sqrt{2}}} \left(\int\limits_{r_0^{-\gamma}}^{\left(\sqrt{2} r_{\mathcal{T}}(t)\right)^{-\gamma}} \left(u^{-\frac{2}{\gamma}} - r_{\mathcal{T}}(t)^2 \right)^{\frac{\eta}{2}} \mathrm{d}u \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}}$$

and for u in this range, $u^{-\frac{2}{\gamma}} \geq 2 r_{\mathcal{T}}(t)^2$ so that

$$\begin{aligned} I_{\gamma,\beta}(S) &\geq C^{-\frac{1}{\gamma}} \left[\int\limits_{\mathcal{T}_{r_0/\sqrt{2}}} \left(\int\limits_{r_0^{-\gamma}}^{\left(\sqrt{2} r_{\mathcal{T}}(t)\right)^{-\gamma}} r_{\mathcal{T}}(t)^\eta \, \mathrm{d}u \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}} \\ &= C^{-\frac{1}{\gamma}} \left[\int\limits_{\mathcal{T}_{r_0/\sqrt{2}}} \left(2^{-\frac{\gamma}{2}} r_{\mathcal{T}}(t)^{\eta-\gamma} - r_0^{-\gamma} r_{\mathcal{T}}(t)^\eta \right)^{\frac{\beta}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}} \\ &\geq C^{-\frac{1}{\gamma}} 2^{-\frac{1}{\gamma}} \left[\int\limits_{\mathcal{T}_{r_0/\sqrt{2}}} r_{\mathcal{T}}(t)^{\frac{\beta(\eta-\gamma)}{\gamma}} \mathrm{d}t \right]^{\frac{1}{\beta}} = \infty \end{aligned}$$

from Lemma 3.2 as it follows from (35) that $(\beta\gamma - \beta\eta)/\gamma > 1 - \dim_{LB}(\mathcal{T})$. $\square$

Note that the conditions are related by the implications $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$ for finite γ, β, so the condition (iii) is sufficient for finite γ, β.

3.2. Examples

We compute the r-codimension print for some subsets of $\mathbb{R}^n \times [0,T]$. While the calculations are straightforward, we find that computing the r-codimension print of even the most elementary subset is quite involved. Fortunately, the result of Theorem 3.4 greatly simplifies these calculations.

Example 3.5. *The singleton set $S = \{0\} \subset \mathbb{R}^n \times [0,T]$ has r-codimension print the union*

$$\begin{aligned}\operatorname{print}_r(S) = &\{(\gamma,\beta) \,|\gamma\beta < \gamma + \beta n \quad 0<\gamma,\beta<\infty\} \\ &\cup \{(\gamma,\infty), 0<\gamma<n\} \cup \{(\infty,\beta), 0<\beta<1\},\end{aligned}$$

illustrated in Figure 1.

Indeed, as S can be written as the product set $\{0\}\times\{0\}$ and $\dim_B(\{0\}) = 0$ conditions (i), (ii) *and* (iii) *of Theorem 3.4 guarantee that the print contains this union. Further, as $\dim_{LB}(\{0\}) = 0$, condition (iv) guarantees that no point of $\{(\gamma,\beta)\,|\gamma\beta > \gamma+\beta n \quad 0<\gamma,\beta<\infty\}$ is in the print.*

In this case, Theorem 3.4 yields the majority of the structure of $\operatorname{print}_r(S)$ *as only the borderline cases remain: we now show that points on the hyperbola $\gamma\beta = \gamma + \beta n$, the points (γ,∞) for $\gamma \geq n$ and the points (∞,β) for $\beta \geq 1$ are not in* $\operatorname{print}_r(S)$*:*

For simplicity we assume that $T \geq \sqrt{2}$. The distance function is given by $r_S(x,t) = \sqrt{|x|^2 + |t|^2}$ and by taking $r_0 = \sqrt{2}$ the rectangular set

$$[-1,1]^n \times [0,1] \subset S_{r_0}.$$

Consequently, by reducing the domain of integration, for $0<\gamma,\beta<\infty$ such that $\gamma\beta = \gamma+\beta n$

$$\begin{aligned}I_{\gamma,\beta}(S) &\geq \left[\int_0^1 \left(\int_{\{x||x|<|t|\}} \left(|x|^2+|t|^2\right)^{-\frac{\gamma}{2}} \mathrm{d}x\right)^{\frac{\beta}{\gamma}} \mathrm{d}t\right]^{\frac{1}{\beta}} \\ &\geq \left[\int_0^1 \left(\int_{\{x||x|<|t|\}} \left(2\,|t|^2\right)^{-\frac{\gamma}{2}} \mathrm{d}x\right)^{\frac{\beta}{\gamma}} \mathrm{d}t\right]^{\frac{1}{\beta}} \\ &\geq \left[\int_0^1 2^{-\frac{\beta}{2}}\,|t|^{-\beta}\,\mathcal{L}^n\{x|\,|x|<|t|\}^{\frac{\beta}{\gamma}} \mathrm{d}t\right]^{\frac{1}{\beta}}\end{aligned}$$

which, as $\mathcal{L}^n\{x|\,|x|<|t|\}=\omega_n\,|t|^n$ *where* ω_n *is the volume of the unit ball in* $\mathbb{R}^n$*,*

$$= \left[\omega_n^{\frac{\beta}{\gamma}} 2^{-\frac{\beta}{2}} \int_0^1 |t|^{n\frac{\beta}{\gamma}-\beta}\,\mathrm{d}t\right]^{\frac{1}{\beta}}$$
$$= \left[\omega_n^{\frac{\beta}{\gamma}} 2^{-\frac{\beta}{2}} \int_0^1 |t|^{-1}\,\mathrm{d}t\right]^{\frac{1}{\beta}}$$

which diverges, so $(\gamma,\beta)\notin \mathrm{print}_r\,(S)$*.*

Next, as $\sup_{\{x|r_S(x,t)<r_0\}}\left(|x|^2+|t|^2\right)^{-\frac{1}{2}}=|t|^{-1}$*,*

$$I_{\infty,1}\,(S) = \int_0^{\sqrt{2}} \operatorname*{ess\,sup}_{\{x|r_S(x,t)<r_0\}} \left(|x|^2+|t|^2\right)^{-\frac{1}{2}}\mathrm{d}t$$
$$= \int_0^{\sqrt{2}} |t|^{-1}\,\mathrm{d}t$$

which diverges, so $(\infty,1)\notin \mathrm{print}_r\,(S)$*. Consequently, from property* (viii) *of Lemma 3.1,* $(\infty,\beta)\notin \mathrm{print}_r\,(S)$ *for all* $\beta\geq 1$*.*

Finally, the domain $\left\{x\middle|\,|x|^2+|t|^2<r_0^2\right\}$ *and the integrand* $\left(|x|^2+|t|^2\right)^{-\frac{n}{2}}$ *are both largest at* $t=0$ *so we clearly have*

$$I_{1,\infty}\,(S) = \operatorname*{ess\,sup}_{t\in[0,T]} \int_{\left\{x\middle||x|^2+|t|^2<r_0^2\right\}} \left(|x|^2+|t|^2\right)^{-\frac{n}{2}}\mathrm{d}x$$
$$= \int_{\{x||x|<r_0\}} |x|^{-n}\,\mathrm{d}x$$

which diverges, so $(n,\infty)\notin \mathrm{print}_r\,(S)$*. Again, property* (viii) *of Lemma 3.1 yields* $(\gamma,\infty)\notin \mathrm{print}_r\,(S)$ *for all* $\gamma\geq n$*.*

In the following example we demonstrate that Theorem 3.4 does not necessarily capture the entire r-codimension print, even for product sets:

Example 3.6. *Let* $A\subset\mathbb{R}^n$ *and* $\mathcal{T}\subset[0,T]$ *be such that*

$$\dim_B\,(A\times\mathcal{T}) < \dim_B\,(A) + \dim_B\,(\mathcal{T})$$

(see Robinson and Sharples (2012) for an example of such sets). From this inequality there exists γ *for which*

$$n+1-\dim_B\,(A)-\dim_B\,(\mathcal{T}) < \gamma < n+1-\dim_B\,(A\times\mathcal{T})$$

Consequently, the point (γ, γ) is in the print of S from Lemma 3.2. However this point is not captured by Theorem 3.4 as

$$\gamma^2 \geq \gamma \left(n - \dim_B (A)\right) + \gamma \left(1 - \dim_B (\mathcal{T})\right)$$

and so does not satisfy condition (iii).

4. Conclusion

Theorem 2.3 allows us to determine if a generalised flow solution of the non-autonomous ODE (1) avoids a specified subset $S \subset \mathbb{R}^n \times [0, T]$ knowing nothing more than the regularity of f and the anisotropic detail of S encoded in its r-codimension print. Although calculating the r-codimension print if a set S is quite involved, a large amount of the structure of the print can be determined from the box-counting dimensions of S and its projections. By combining this geometric result with the avoidance criterion we arrive at the following corollary:

Corollary 4.1. *Let X be a generalised flow solution of the ODE* (1) *with vector field $f \in L^p(0, T; L^q(\mathbb{R}^n))$ for some $1 \leq p, q \leq \infty$. If S is a compact subset of $\mathbb{R}^n \times [0, T]$ such that at least one of*

(i) $q^* < n - \dim_B (P_x(S))$,

(ii) $p^* < 1 - \dim_B (P_t(S))$, *or*

(iii) $q^* p^* < q^* \left(1 - \dim_B (P_t(S))\right) + p^* \left(n - \dim_B (P_x(S))\right)$,

holds, where $P_t(S) \subset [0, T]$ is the projection of S onto the temporal component and $P_x(S) \subset \mathbb{R}^n$ is the projection of S onto the spatial component, then the flow X avoids the subset S.

We remark that the condition (ii) is vacuous as the Hölder conjugate p^* is not less than 1.

This corollary gives sufficient but not necessary conditions for avoidance. Indeed, for each $\varepsilon \in (0, 1)$ we can find[3] a closed set $A \subset \mathbb{R}^n$ with $\dim_B (A) = n - \varepsilon$ and a countable, closed set $\mathcal{T} \subset [0, T]$ with $\dim_B (\mathcal{T}) = 1 - \varepsilon$

[3]For example let A be a generalised Cantor set (see Robinson and Sharples (2012)) and let $\mathcal{T} := \{0\} \bigcup_{k=1}^{\infty} \{k^{-\alpha}\}$ with $\alpha = \varepsilon / (1 - \varepsilon)$ (see Example 13.4 of Robinson (2001)).

and consider the product $S := A \times [0,T]$. Clearly for a given vector field f we can choose ε sufficiently small that none of the conditions (i)-(iii) are satisfied. However, it follows from Lemma 2.1 that every generalised flow solution avoids the set S as $\mathcal{T}$ is countable and A has zero Lebesgue measure (the Hausdorff dimension of A satisfies $\dim_H(A) \leq \dim_B(A) < n$, see Falconer (2003) pp.46).

In Section 1.6 we saw that it is possible to use the avoidance property to determine stronger uniqueness properties for irregular ODEs than those currently provided by the general theory of DiPerna & Lions. Further, in Robinson and Sadowski (2009) the avoidance property is used to give improved regularity of the flow: in this case the generalised flow avoids the discontinuities of f so almost every trajectory is continuously differentiable in time. It would be interesting to see if geometric tools such as the avoidance property can further supplement the powerful functional analytic approach to irregular ODEs of DiPerna & Lions.

Finally, it is of interest to determine if the avoidance criterion of Theorem 2.3 is sharp by attempting to produce examples similar to those in Aizenman (1978a), as described in Section 1.6. While Aizenman's autonomous examples could simply be recast as non-autonomous ODEs, it would be interesting to find similar borderline cases for vector fields with arbitrary temporal regularity.

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