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Fractal property of generalized M-set with rational number exponent

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ABSTRACT

Dynamic systems described by $f_c(z) = z^2 + c$ is called Mandelbrot set (M-set), which is important for fractal and chaos theories due to its simple expression and complex structure. $f_c(z) = z^k + c$ is called generalized M set (k–M set). This paper proposes a new theory to compute the higher and lower bounds of generalized M set while exponent k is rational, and proves relevant properties, such as that generalized M set could cover whole complex number plane when $k < 1$, and that boundary of generalized M set ranges from complex number plane to circle with radius 1 when k ranges from 1 to infinite large. This paper explores fractal characteristics of generalized M set, such as that the boundary of k–M set is determined by k, when $k = p/q$, where p and q are irreducible integers, (GCD(p,q) = 1, $k > 1$), and that k–M set can be divided into $|p-q|$ isomorphic parts.

1. Introduction

The dynamic system $f_c(z) = z^2 + c$ is called Mandelbrot set (M-set) [1] as Mandelbrot first proposed it in 1982. Later, Gujar and Dhurandhar extended M-set into generalized M-set $f_c(z) = z^k + c$ with different exponent k. When exponent is k, it is called k–M set. Thus we can see that original M-set is 2-M set.

k–M set is important for chaos and fractal theories. Many scholars have conducted researches on k–M sets. Huang studied periodic orbit of k–M set when k is a positive integer [2]. Gujar and Dhurandhar explored the structure of k–M set where k is a real number [3,4]. Shirriff researched characteristics of k–M set where k is a complex number [5].

After year 2000, more progresses have been made with advancement of computer techniques, especially computer graphic techniques. Wang explored escape time of M-set [6]. Noah analyzed the radius of M-set [7]. Pastor and Ashish analyzed chaotic features of M-set [8–9]. Fractal is widely used with computer techniques [10–12,14] when M-set continues to attract researchers' attentions [13].

However, there are still some problems unsolved. For example, the structure of k–M set is unknown when k is not an integer.

This paper presents our following results in research of k–M set where k is a rational number:

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(i) Boundary of k–M set falls between circle with radius $R = (k-1)\cdot k^{-\frac{k}{k-1}}$ and circle with radius $R = 2^{\frac{1}{k-1}}$ when $k > 1$.
(ii) A k–M set could contain whole complex number plane when $k < 1$.
(iii) If $k = p/q$, p and q are irreducible integers, (GCD(p,q) = 1, $k > 1$); then the symmetry property of k–M set is decided by p and q.

The remainder of the paper is organized as follows. We analyze the boundary of k–M set with $k > 0$ in Section 2. Then, we analyze the boundary of k–M set with $k < 0$ in Section 3. Then, we analyze symmetry property of k–M Set with exponent k when $k > 1$ in Section 4. Finally, Section 5 summarizes the main results of the paper.

2. Boundary of *k*–*M* set with exponent *k* > 0

Falconer [14] gives the following extended M Set definition:

Definition 1. Let $f_c^0(0) = c, f_c^i(0) = \left(f_c^{i-1}(0)\right)^2 + c$, where i is a natural number. The M set is defined by Eq. (1):

$$M = \left\{c \in \mathbb{C} : f_c^i(0) \nrightarrow \infty \quad \text{when} \;\; i \to \infty\right\}. \tag{1}$$

A definition of k–M set can be generalized from above M set definition:

Definition 2. Let $f_c^0(0) = c, f_c^i(0) = \left(f_c^{i-1}(0)\right)^k + c$, where i is a natural number. The k–M set is defined by Eq. (2):

$$k\text{—}M \;\; \text{set} = \{c \in \mathbb{C} : f_c^i(0) \nrightarrow \infty \quad \text{when} \;\; i \to \infty\}. \tag{2}$$

Lemma 1. *Let* $0 < k < 1$, *all positive real numbers are in k–M set.*

Proof. When $0 < k < 1$ and $x > 0$, $x^k < x \Longleftrightarrow x > 1$, $x^k > x \Longleftrightarrow x < 1$, $x^k = x \Longleftrightarrow x = 1$.

In the following paragraphs, these three cases are analyzed.

(i) Case when $x > 1$

Let $f(x) = x^k$ and $g(x) = kx + 1 - k$. We have $f(1) = g(1) = 1$ as $1 = 1^k = k\cdot 1 + 1 - k$, and $f'(x) < g'(x)$ as $kx^{k-1} < k$ because when $x > 1$, and $0 < k < 1$, $x^{k-1} < 1$.

Hence, $f(x) < g(x)$ when $x > 1$, i.e., $x^k < kx + 1 - k$ when $x > 1$.

The following proof is based on mathematical induction.

First, let $i = 1$, we have

$$1 < f_x^1(0) = x + x^k < x + kx + 1 - k = x\cdot\sum_{p=0}^{1}(k^p) + (1 - k^1) \quad (x > 1,\;\; 0 < k < 1). \tag{3}$$

Then, assume when $i \leqslant j$, we have

$$1 < f_x^j(0) \leqslant x\cdot\sum_{p=0}^{j}(k^p) + (1 - k^j). \tag{4}$$

Hence, let $i = j + 1$, we have

$$\begin{aligned} f_x^{j+1}(0) &= \left(f_x^j(0)\right)^k + x \leqslant \left(x\cdot\sum_{p=0}^{j}(k^p) + (1-k^j)\right)^k + x \leqslant k\left(x\cdot\sum_{p=0}^{j}(k^p) + (1-k^j)\right) + 1 - k + x \\ &= x\cdot\sum_{p=0}^{j+1}(k^p) + (1 - k^{j+1}). \end{aligned} \tag{5}$$

From Eqs. (3)–(5), we know $f_x^j(0) \leqslant x\cdot\sum_{i=0}^{j}(k^i) + (1-k^j) = x\frac{1-k^{j+1}}{1-k} + 1 - k^j$ for every iteration time j.

Then, $\lim_{j\to\infty} f_x^j(0) < \frac{x}{1-k} + 1 \nrightarrow \infty$. Thus we can see that x is in k–M set.

(ii) Case when $0 < x < 1$

From $f_x^1(0) = x + x^k < x + 1$, we know $f_x^{j+1}(0) < f_x^j(x+1) < f_{x+1}^{j+1}(0)$. Then we know $f_{x+1}^{j+1}(0)$ is not divergent from case (i). Hence, $f_x^{j+1}(0)$ is not divergent. Thus, x is in k–M set.

(iii) Case when $x = 1$

$f_1^j(0) = \left(\cdots(1+1)^k\cdots\right)^k + 1 = f_1^{j-1}(2)$. $f_1^j(2) < f_2^j(2) = f_2^{j+1}(0)$. Thus it is convergent by case (i). Hence, $x = 1$ is in k–M set.

From cases (i), (ii), (iii), Lemma 1 is proved. □

Lemma 1 shows that all positive real numbers are in k–M set when $0 < k < 1$ and k is a rational number. The following Lemma 2 and Inference 1 extend the result to whole complex number plane.

Lemma 2. *For any complex number z and natural number* $n = 1,2,\ldots, |f_z^n(0)| \leqslant |f_c^n(0)| \quad \text{when } |z| = c, \text{ for any } k\text{—}M \text{ set.}$

Proof. Considering any point z in the complex plane, we assume $|z| = c$.

Then, $|f_z^i(0)| \leqslant |f_c^i(0)|$ can be proved by mathematical induction.

Let $i = 0$, we have

$$|f_z^0(0)| = |z| \leqslant c = |f_c^0(0)|. \tag{6}$$

Then, assume when $i \leqslant n$, we have

$$|f_z^n(0)| \leqslant |f_c^n(0)|. \tag{7}$$

Hence, let $i = n + 1$, we have

$$|f_z^{n+1}(0)| = \left|(f_z^n(0))^k + z\right| \leqslant |f_z^n(0)|^k + |z| \leqslant |f_c^n(0)|^k + c = |f_c^{n+1}(0)|. \tag{8}$$

From Eqs. (6)–(8), Lemma 2 is proved. □

The following Inference 1 extends Lemma 1 to complex number plane.

Inference 1. Let $0 < k < 1$ and k is a rational number, all points in complex number plane are in k–M set.

Proof. We assume z as any complex number and $|z| = c$.

From Lemma 1, c is in k–M set as c is a positive real number. When $f_c^{n+1}(0)$ is convergent, $f_z^{n+1}(0)$ is convergent by Lemma 2. Hence, z is in k–M set for any point z in complex plane. □

When $k = 1$ and $x \neq 0$, $f_x^j(0) = x(j+1)$; when $j \to \infty$, $f_x^j(0) \to \infty$. Hence, k–M set contains only point zero when $k = 1$.
Now, we analyze k–M set when $k > 1$.

Lemma 3. *Let $k > 1$ and k is a rational number. k–M set is bounded and the minimum absolute value of points at boundary is* $((k-1) \cdot k^{-\frac{k}{k-1}}$.

Proof. The minimum absolute value c of points at boundary must satisfy equation $x^k + c = x$, where x is extremum of sequence of numbers defined by $f_c^0(0) = c, f_c^i(0) = \left(f_c^{i-1}(0)\right)^k + c$.First, we prove that $c = (k-1) \cdot k^{-\frac{k}{k-1}}$ is in k–M set.When $k > 1$, consider function $g(x) = x - x^k$. Let the derivative $g'(x) = 1 - kx^{k-1} = 0$, we can see that $x = k^{-\frac{1}{k-1}}$ is the extremum of $g(x)$. So we know that $c = g(x) = x(1 - x^{k-1}) = (k-1)k^{-\frac{k}{k-1}}$.Let

$$c = (k-1) \cdot k^{-\frac{k}{k-1}} \tag{9}$$

and

$$x = k^{-\frac{1}{k-1}}. \tag{10}$$

We obtain Eq. (11) using Eqs. (9) and (10).

$$f_c^1(x) = x^k + c = x. \tag{11}$$

Then, from $f_c^2(x) = \left(f_c^1(x)\right)^k + c = x^k + c = x$, we have $f_c^i(x) = x$. Hence, $f_c^i(0) < f_c^i(x) = f_c^1(x) = x$. It means that c is in k–M set. Then, from Lemma 2, we know that any complex number z is in k–M set when $|z| < c$.Then we prove that c is the lower bound value, which means that there is at least one point z that is not in k–M set with $|z| > c$.Let $c^* = c + \omega_1 > c$ and there exists a positive integer n that makes $f_{c^*}^n(0) = x + \omega_2 > x$.To use Eq. (11) and $kx^{k-1} = 1$, we prove it as follows.Let $i = n + 1$,

$$f_{c^*}^{n+1}(0) = (x + \omega_2)^k + c + \omega_1 \geqslant x^k + c + \omega_1 = x + \omega_1 \cdot 1. \tag{12}$$

Then, let $i = n + m$, assuming

$$f_{c^*}^{m+n}(0) \geqslant x + \omega_1 \cdot m. \tag{13}$$

Hence, let $i = n + m + 1$, we have

$$f_{c^*}^{m+n+1}(0) = \left(f_{c^*}^{m+n}(0)\right)^k + c + \omega_1 \geqslant (x + \omega_1 m)^k + c + \omega_1 \geqslant x^k + kx^{k-1}\omega_1 m + c + \omega_1 = x + \omega_1(m+1). \tag{14}$$

To conclude Eqs. (12)–(14), we know when $i \to \infty$, $m \to \infty$, $x + \omega_1 \cdot m \to \infty$, $f_{c^*}^i(0) \to \infty$.It means c^* is not in k–M set, hence, c is the minimum boundary point.Lemma 3 is proved. □

Furthermore, considering process of proof in Lemma 3, we affirm that point z at boundary with the maximum absolute value c must satisfy $z' = ze^{\frac{2n\pi i}{k}} = z^k + z$. In other words, function f maps z to z' with same module and different modular phase angle. Thus, it means that we can calculate z from the equation. So we reach Lemma 4.

Lemma 4. *Let $k > 1$. The maximum absolute value of boundary points is no more than ($\leqslant$) $2^{\frac{1}{k-1}}$.*

Proof. At first, we prove $2^{\frac{1}{k-1}}$ is a threshold of bound.

The solutions c of Eq. (15) are at boundary because $f_c^i(0) = f_c^1(0)$ when $i > 1$.

$$c \cdot e^{\frac{2n\pi i}{k}} = c^k + c. \tag{15}$$

Then we solve $c \cdot e^{\frac{2n\pi i}{k}} = c^k + c$ with $c \neq 0$ and find the solutions are $c = \left(e^{\frac{2n\pi i}{k}} - 1\right)^{\frac{1}{k-1}}$. So we have Eq. (16).

$$|c| = \left|e^{\frac{2n\pi i}{k}} - 1\right|^{\frac{1}{k-1}} \leqslant 2^{\frac{1}{k-1}} (n \in N). \tag{16}$$

We assume that k=p/q is irreducible, it makes $e^{\frac{2n\pi i}{k}} = e^{\frac{2qn\pi i}{p}}$. So when p is even, let n=p/2, we have $e^{\frac{2n\pi i}{k}} = e^{q\pi i}$. Then, let q is odd, we have $e^{q\pi i}$=-1 and |c|=$2^{\frac{1}{k-1}}$. On opposite side, when p is odd, we know there exist u and v make pu+qv=1. So when we set n=vn_1, we find Eq. (17).

$$\begin{aligned} &2qn = 2qvn_1 = 2n_1(1 - pu), \\ &\frac{2qn}{p} = \frac{2n_1}{p} - 2un_1. \end{aligned} \tag{17}$$

Then we reach Eq. (18) from Eq. (17).

$$e^{\frac{2n\pi i}{k}} = e^{\frac{2qn\pi i}{p}} = e^{\left(\frac{2n_1}{p} - 2un_1\right)} = e^{\frac{2n_1\pi i}{p}}. \tag{18}$$

In other words, when $n_1 = \frac{p-1}{2}$ *or* $\frac{p+1}{2}$ as well as $n = v\frac{p\pm1}{2}$, we find Eq. (19)

$$\text{Solution}_{\max} = 2^{\frac{1}{k-1}} \cdot \left(\sin\frac{p-1}{2p}\pi\right)^{\frac{1}{k-1}}. \tag{19}$$

Then, we prove that a point z with $|z| > 2^{\frac{1}{k-1}}$ is not in k–M set.

When we assume $|z| = 2^{\frac{1}{k-1}} +$ ($\varepsilon > 0$), we use mathematical induction to prove it.

Let $i = 0$, we get Eq. (20).

$$|f_z^0(0)| = |z| \geqslant 2^{\frac{1}{k-1}} + \varepsilon \cdot (2k-1)^0. \tag{20}$$

Then, let $i = j$, assuming

$$|f_z^j(0)| \geqslant 2^{\frac{1}{k-1}} + \varepsilon \cdot (2k-1)^j. \tag{21}$$

Hence, let $i = j + 1$, we reach Eq. (22)

$$|f_z^{j+1}(0)| = |\left(f_z^j(0)\right)^k + z| \geqslant |f_z^j(0)|^k - |z| \geqslant \left(2^{\frac{1}{k-1}} + \varepsilon \cdot (2k-1)^j\right)^k - 2^{\frac{1}{k-1}} - \varepsilon. \tag{22}$$

Using Newton Binomial theorem, we find Eq. (23). In this formula, $R > 0$ is remainder term.

$$\begin{aligned} \left(2^{\frac{1}{k+1}} + \varepsilon \cdot (2k-1)^i\right)^k - 2^{\frac{1}{k-1}} - \varepsilon &= \left(2^{\frac{k}{k-1}} - 2^{\frac{1}{k-1}}\right) + 2k\varepsilon \cdot (2k-1)^i - \varepsilon + R \geqslant 2^{\frac{1}{k-1}} + 2k\varepsilon \cdot (2k-l)^i - \varepsilon \\ &\geqslant 2^{\frac{1}{k-1}} + 2k\varepsilon \cdot (2k-1)^i - \varepsilon \cdot (2k-1)^i = 2^{\frac{1}{k-1}} + \varepsilon \cdot (2k-1)^{i+1}. \end{aligned} \tag{23}$$

When applying Eq. (23) into Eq. (22), we have Eq. (24).

$$|f_z^{j+1}(0)| \geqslant 2^{\frac{1}{k-1}} + \varepsilon \cdot (2k-1)^{i+1}. \tag{24}$$

From Eqs. (21), (22), and (24), we have Eq. (25).

$$\lim_{i\to\infty} |f_z^i(0)| \geqslant 2^{\frac{1}{k-1}} + \varepsilon \cdot \lim_{i\to\infty} (2k-1)^i = \infty (2k-1 > 1). \tag{25}$$

It means z is out of k–M set.

To conclude, Lemma 4 is proved. □

Then we gain Theorem 1 from Lemmas 1–4 and Inference 1. Theorem 1 is boundary conclusion of *k–M* set with exponent k > 0.

Theorem 1. *Domain of k–M set with exponent k depends on k by k > 0 as below:*

(a) *Domain of k–M set contains complex plane when 0 < k < 1.*
(b) *Domain of k–M set contains only point origin (zero) when k = 1.*
(c) *To set point origin center point, maximum radius of k–M set is* $R_{\max} = 2^{\frac{1}{k-1}}$ when $k > 1$.
(d) *To set point origin center point, minimum radius of k–M set is* $R_{\min} = (k-1) \cdot k^{-\frac{k}{k-1}}$ *when k > 1.*

Items in Theorem 1 show domain and radius of *k–M* set. These items are all proved in lemmas and inference above. Then we can find Inference 2 from Theorem 1.

Inference 2. The extremum of *k–M* set is a circle with center origin and radius 1 when exponent $k \to \infty$.

Proof. We know $\lim_{k\to\infty} 2^{\frac{1}{k-1}} = \lim_{n\to\infty}(k-1)\cdot k^{-\frac{k}{k-1}} = 1$. Moreover, $(k-1)\cdot k^{-\frac{k}{k-1}} < 1 < 2^{\frac{1}{k-1}}$. So Inference 2 is proved. □

This is results of domain and bound of *k–M* set when k > 0. Later we will study *k–M* set with $k < 0$ in Section 3.

3. Boundary of *k–M* set with exponent k (k < 0)

Similar to proof in Section 2, we find bound of *k–M* set by considering k in two ranges, which are $k < -1$ and $-1 \leqslant k < 0$. When $k < -1$, we present Lemma 5 to show convergence about *k–M* set.

Lemma 5. *Let* $k \leqslant -1$, *all positive real numbers are in k–M set.*

Proof

(i) $c \geqslant 1$
It is known that $0 < c^k \leqslant 1$ when $c > 1$. Then, when we see $1 < c < x^k + c < c + 1$ when $x > 1$ and $c > 1$, we know that $f_c^{i+1}(0) = \left(f_c^i(0)\right)^k + c < c + 1$. Moreover, $f_1^i(0) = f_1^{i-1}(2) > 1$, which means that c is in *k–M* set when $c \geqslant 1$.
(ii) $0 < c < 1$
When we assume that there exists i making $f_c^i(0) > 1$, we find $f_c^{i+1} < c + 1$. It means c is in *k–M* set. When there does not exist i making $f_c^i(0) > 1$, besides $f_c^i(0) > 0$, we know that c is also in *k–M* set.

Summarizing cases (i) and (ii), Lemma 5 is proved. □
So with the similar idea in Inference 1, we have Inference 3 that applied to all points in complex plane.

Inference 3. Let $k < -1$, *k–M* set contains all complex plane.

Proof. As we known, $f_c^i(0)$ is finite when c is a positive integer. We set $|f_z^i(0)| = c_i$ and find Eq. (26).

$$c_{i+1} = |f_z^{i+1}(0)| = |\left(f_z^i(0)\right)^k + z| \leqslant |f_z^i(0)|^k + |z| = c_i^k + c_1. \tag{26}$$

So we know that $\lim_{i\to\infty} c_i \neq \infty$. Otherwise, $\lim_{i\to\infty} c_{i+1} \leqslant \lim_{i\to\infty} c_i^k + c_1 = c_1 \neq \infty$. It is based on reduction to absurdity.
So we know that Inference 3 is proved. □

Then we study *k–M* set with $-1 \leqslant k < 0$. At first, we know that $c_{i+1} \leqslant c_i^k + c_1$ from Inference 2 by considering $|f_z^i(0)| = c_i$. When we study it with similar idea in inference 2, we reach Inference 4. It shows that *k–M* set contains all complex plane when $-1 \leqslant k < 0$.

Inference 4. Let $-1 \leqslant k < 0$, *k–M* set contains all complex plane.

So we find that *k–M* set contains whole complex plane when $k < 0$ from Lemma 4 and Inference 2 and 3. It means that symmetry characteristics we have to study only apply to case $k > 1$ of *k–M* set. Next we will find some symmetry characteristics of *k–M* set with $k > 1$ in Section 4.

4. Symmetry characteristics of *k–M* set with exponent k (k>1)

As we know, fractals of *k–M* set with integer exponent n, which is called *n–M*, show their symmetry by $n > 2$. But there is no proved conclusion about *k–M* set when exponent *k* is not integer. In this section, we will prove the symmetry and other fractal characteristics of *k–M* set when *k* is positive.

Firstly, we know that *k–M* set are axial symmetrical about real axis because $\bar{z}^k + \bar{C} = \overline{Z^k} + \bar{C} = \overline{Z^k + C}$ for any *z* and *c* in complex plane. Then, when rational number $k = p/q$ (*p* and *q* are irreducible), we reach Theorem 2.

Theorem 2. *Number of isomorphic subset clusters in k–M set is p-q.*

Proof. At first, we explain that when we say *k–M* set has *p–q* isomorphic subset clusters, if and only if, we prove all points *z* in one subset clusters of *k–M* set, there are *p–q–*1 other points z_i in other *p–q–*1 subset clusters with same modular in *k–M* set which are iterated to same modular and phase angle except origin in complex plane. In other words, if $|f_z^p(0)| = |f_{z_1}^p(0)|$ and $f_z^p(0)$ has same different phase angle to $f_{z_1}^p(0)$, we call *z* and z_i are isomorphic. Isomorphic clusters are make up of all isomorphic points.

For each point $z = r \cdot e^{\theta i}$ in *k–M* set, if there exist $z_1 = r \cdot e^{\theta_1 i}$ makes $|f_z^p(0)| = |f_{z_1}^p(0)|$ when p>0, it means that we find an isomorphic point of z. To solve this equation, we first solve equation $|z^k + z| = |z_1^k + z_1|$. To predigest it, we get Eq. (27).

$$r^{k-1} \cdot e^{(k-1)\theta i} = r^{k-1} \cdot e^{(k-1)\theta_1 i}. \tag{27}$$

To solve Eq. (27), we find Eq. (28).

$$\theta = \theta_1 + \frac{2n\pi}{k-1}(n = 1 \sim k-1). \tag{28}$$

Applying k=p/q in Eq. (28), we have Eq. (29).

$$\theta = \theta_1 + \frac{2qn\pi}{p-q}. \tag{29}$$

Furthermore, difference between phase angles is $\theta - \theta_1$.

Then, when we get Eq. (31) from Eq. (30).

$$f_z^p(0) = f_{z_1}^p(0) \cdot e^{\frac{2n\pi}{k-1}i}. \tag{30}$$

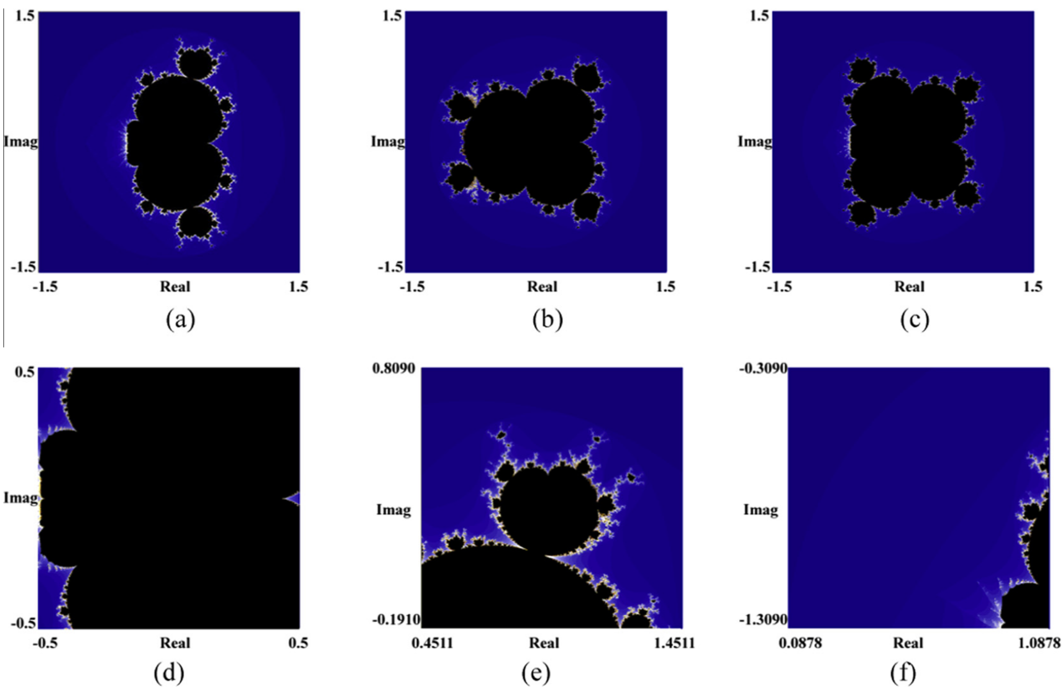

Fig. 1. A drawn *k–M* set by classic method. Parameter *k* in (a)–(c) is 3.5, 4.5 and 5.5. Parameter *k* in (d)–(f) is 3.5. In (a)–(c), displayed areas are 3×3 with center 0. In (d)–(f), displayed areas are 1×1 with center $e^{\frac{0\pi i}{5}}$, $e^{\frac{2\pi i}{5}}$ and $e^{\frac{4\pi i}{5}}$.

$$f_z^{p+1}(0) = \left(f_z^p(0)\right)^k + z = \left(f_{z_1}^p(0)\right)^k \cdot e^{\frac{2kn\pi}{k-1}i} + z_1 \cdot e^{\frac{2n\pi}{k-1}i} = (\left(f_{z_1}^p(0)\right)^k + z_1) \cdot e^{\frac{2n\pi}{k-1}i} = f_{z_1}^{p+1}(0) \cdot e^{\frac{2n\pi}{k-1}i}. \quad (31)$$

It means that our assuming is true. Furthermore, we can calculate the number of isomorphic points of z. To use $k = p/q$ in Eq. (31), we get $z = z_1 \cdot e^{\frac{2qn\pi}{p-q}}$. It means that z and z_i are isomorphic when $n = 1 \sim p\text{–}q$.

Based on all above, Theorem 2 is proved. □

Then, we have to say that there are some minor errors in some thesis that draws fractal figures of k–M set by using computer. The errors are caused by calculation. When we calculate a point z in complex variable function, we also change the point z to complex exponent form that is used to operate. But in calculation, computer will translate it into form $a + bi$ to operate. Then some values would be changed. The difference between the results calculated by these two kinds once is $e^{2m\pi\theta}$, and the results of following mappings will be wrong. For example, when a computer calculates $\mathrm{e}^{(\frac{a\pi}{b})i\cdot\frac{b}{a}}$, the result is correct. However, when it calculates $\left(\mathrm{e}^{(\frac{a\pi}{b})i}\right)^{\frac{b}{a}}$, the result is always wrong. Then, the resulting mapping will be always false.

For example, we use $k = 3.5$ to validate it. In this case, $p = 7$, $q = 2$. So we get minimum and maximum radius are $\frac{5}{7} \cdot \left(\frac{2}{7}\right)^{\frac{2}{5}}$ and $2^{\frac{2}{5}}$. We validate our opinion by use Fig. 1. In Fig. 1a–c, we create some classic k–M set with $p = 7, 9, 11$ and $q = 2$. We can see generation of k–M set by k increase by compare Fig. 1a to b and c. But just in another example in Fig. 1d, e and f, we trust there are $7 - 2 = 5$ isomorphic parts in fractal figure of 3.5-M set. Then we use a point to execute an experiment in Fig. 1. As we seen, a point $z = \mathrm{e}^{\frac{\pi i}{3.5-1}} = e^{\frac{2\pi i}{5}}$ is absolutely in k–M set. Moreover, it is 2-periodic point because $z^{\frac{7}{2}} + z = 0$. So in our idea, we trust $e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}}, e^{2\pi i}$ are all 2-periodic points. But in Fig. 1d–f, which is drawn by classic method, it is not suitable. In fact, center point in 'd' and 'e' are convergence, but 'f' are divergence. It validates our conclusion.

In fact, when we calculate these points in k–M set, we find they are all 2-periodic points. For example, let $z = \mathrm{e}^{1.2\pi i} \approx$ -0.8090-0.5878i, $z^{3.5} + z = z(z^{2.5} + 1) = z(z^{3\pi i} + 1) = 0$, then the other points make $z^{3.5} + z = 0$. This is also a validation of our idea. Additionally, if there are some isomorphic subsets with different period m and n overlapped, it means these parts have both characteristics both of periods m and n. The conclusion is absurd when m and n are irreducible. So we conjecture that all periods are disjunctive in k–M set. This is equivalent to that radii of all periodic areas are less than $\frac{\pi}{p-q}$ when we assume a periodic area is an isomorphic circle.

Conjecture 1. Assuming a periodic area is a isomorphic circle, radii of periodic areas are less than $\frac{\pi}{p-q}$.

Admittedly, our idea considers the complex plane as composed by plane-layers with phase angle range $[2n\pi, 2n\pi + 2\pi)$. When considering complex plane that does not consist of plane-layers with phase angle more than 2π, we should subtract some boundary points from Theorem 2.

5. Conclusion and future work

We analyzed k–M set when k is rational number in this paper. We first computed bounds of k–M set and proved its symmetry is determined by $k = p/q$ when p and q are irreducible. Then, we proved k–M set contains whole complex plane when $k \leqslant 1$. In other words, k–M set has no bound when $k \leqslant 1$. Next, except for some classic figures of k–M set, we found that k–M set can be divided into p–q isomorphic parts. Of course, these parts cannot overlap with each other. Finally, we give a conjecture to find bound of periodic subsets.

Our future work can be conducted in two steps. At first, we will find a new computational algorithm in complex plane. This algorithm is used to find correct results of exponent calculation in complex plane. We will design the algorithm to avoid phase angle errors by considering intermediate results in computing process. Secondly, we will improve our design. We can assume complex plane is constructed by plane layers of phase angle ranges of $[2n\pi, 2n\pi + 2\pi)$ where $n = 1, 2\ldots$ A mapping will be executed from one plane layer to another. All plane-layers form the whole perfect complex plane.

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