On solutions of the transport equation in the presence of singularities.

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Abstract

We consider the transport equation on $[0,T] \times \mathbb{R}^n$ in the situation where the vector field is BV off a set $S \subset [0,T] \times \mathbb{R}^n$. We demonstrate that solutions exist and are unique provided that the set of singularities has a sufficiently small anisotropic fractal dimension and the normal component of the vector field is sufficiently integrable near the singularities. This result improves upon recent results of Ambrosio who requires the vector field to be of bounded variation everywhere.

In addition, we demonstrate that under these conditions almost every trajectory of the associated regular Lagrangian flow does not intersect the set S of singularities.

Finally, we consider the particular case of an initial set of singularities that evolve in time so the singularities consists of curves in the phase space, which is typical in applications such as vortex dynamics. We demonstrate that solutions of the transport equation exist and are unique provided that the box-counting dimension of the singularities is bounded in terms of the Hölder exponent of the curves.

1 Introduction

In this note we are concerned with the existence and uniqueness of solutions to the transport equation

$$\begin{cases} \partial_t u + b \cdot \nabla u = 0 & \text{on } (0,T) \times \mathbb{R}^n \\ \qquad u(0,\cdot) = u_0(\cdot) \end{cases} \tag{TE}$$

for some $T > 0$, when the non-autonomous vector field $b\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ has limited regularity. Classically, the existence of unique smooth solutions to (TE) is assured if the vector field b and the initial data u_0 are both smooth. However, in many applications such as Fluid Dynamics or Control Theory the smoothness or even continuity of vector fields cannot be guaranteed.

When considering less regular vector fields b, minimally requiring that both b and its (spatial) divergence are locally integrable, we say that a bounded map u is a weak solution of the transport equation if (TE) holds distributionally:

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Definition 1.1. *A map* $u \in L^\infty_{\text{loc}}\left((0,T)\times\mathbb{R}^n\right)$ *is a weak solution of* (TE) *with initial data* $u_0 \in L^\infty_{\text{loc}}\left(\mathbb{R}^n\right)$ *if*

$$\int_{\mathbb{R}^n} u_0\left(x\right)\phi\left(0,x\right)\mathrm{d}x + \int_0^T\int_{\mathbb{R}^n} u\cdot\left(\frac{\partial\phi}{\partial t} + b\cdot\nabla\phi + \operatorname{div} b\ \phi\right)\,\mathrm{d}x\ \mathrm{d}t = 0 \tag{1}$$

for all test maps $\phi \in C^\infty_c\left([0,T)\times\mathbb{R}^n\right)$.

The transport equation (TE) corresponds with the ordinary differential equation

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = b\left(t,\xi\right) \qquad\qquad \xi\left(0\right) = x. \tag{ODE}$$

Classically, solutions of (TE) are obtained via the 'method of characteristics' where the initial data u_0 is evolved along the flow solution of (ODE).

In the less regular setting first considered by DiPerna & Lions [17] this correspondence reverses: solutions of (ODE) are obtained from solutions of (TE). Here the appropriately weakened notion of a flow solution is that of a regular Lagrangian Flow (see DiPerna & Lions [17], Ambrosio [2] and Crippa & De Lellis [11]).

Definition 1.2. *A map* $X\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ *is a* regular Lagrangian Flow solution *of* (ODE) *if*

- *for almost every* $x\in\mathbb{R}^n$ *the trajectory* $t\mapsto X\left(t,x\right)$ *is absolutely continuous and*
$$X\left(t,x\right) = x + \int_0^t b\left(\tau, X\left(\tau,x\right)\right)\mathrm{d}\tau, \tag{2}$$
and

- *there exists a constant* $L>0$ *such that for all Borel sets* $B\subset\mathbb{R}^n$ *the image measure*
$$\mu_n\left(X\left(t,\cdot\right)^{-1}\left(B\right)\right) \le L\mu_n\left(B\right) \qquad\qquad \forall t\in[0,T] \tag{3}$$
where μ_n *is the* n*-dimensional Lebesgue measure.*

As the trajectories are absolutely continuous the integral equality (2) is equivalent to requiring $X\left(0,x\right) = x$ and $\frac{\mathrm{d}X}{\mathrm{d}t} = b\left(t, X\left(t,x\right)\right)$ for almost every $t\in[0,T]$. Further, the condition (3) ensures that sets with positive measure do not evolve into sets with zero measure. From this fact it follows that X is also a regular Lagrangian Flow solution of (ODE) for all vector fields $\tilde{b}$ that are equal to b almost everywhere.

Solutions of (ODE) are obtained from solutions of (TE) via a 'reverse method of characteristics': the existence, uniqueness and stability of regular Lagrangian flows follows from the existence, uniqueness and stability of weak solutions to (TE). We refer to DiPerna & Lions [17] for the original proofs, Ambrosio [2, 3] for a more general approach, or Crippa [10] or De Lellis [15] for a more direct treatment in the case when b is bounded.

Crippa & De Lellis [11] provide an alternative approach to establishing existence and uniqueness of regular Lagrangian flows. The authors use the theory of maximal functions to obtain some new estimates on flows with Sobolev regularity, thereby obtaining uniqueness directly in the ODE framework. However, this approach requires slightly stronger regularity assumptions for the vector field b than the one considered by Ambrosio [2, 3]. We mention that a recent paper by Nguyen [26] provides well-posedness of the flow associated to vector fields represented as singular integrals of BV functions, by extending the estimates on the flow established by Crippa & De Lellis [11]. In turn, this implies well-posedness of the corresponding continuity and transport equations.

In this note we extend the theory of DiPerna & Lions [17] and Ambrosio [2] to vector fields b that are BV off a set $S \subset [0,T] \times \mathbb{R}^n$. Let $d_S \colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be the Euclidean distance function to S, that is

$$d_S(t,x) := \inf_{(s,y)\in S} |(t,x) - (s,y)|,$$

for all $\varepsilon > 0$ we write $\{d_S > \varepsilon\} := \{(t,x) \ : \ d_S(t,x) > \varepsilon\}$ and define sets with corresponding inequalities similarly.

Our main result is the following:

Theorem 1.3. *Let $S \subset [0,T] \times \mathbb{R}^n$ be compact. If the vector field b satisfies*

i) $b \in L^1_{\text{loc}}((0,T) \times \mathbb{R}^n)$,

ii) $\operatorname{div} b \in L^1(0,T; L^\infty(\mathbb{R}^n))$,

iii) $\frac{b}{1+|x|} \in L^1((0,T) \times \mathbb{R}^n) + L^1(0,T; L^\infty(\mathbb{R}^n))$,

iv) *for all $\Omega \subset\subset S^c$, the restriction $b|_\Omega \in L^1(0,T; BV_{\text{loc}}(\mathbb{R}^n))$,*

v) *for some $1 \le p, q \le \infty$*

$$b \cdot \nabla d_S \in L^p(0,T; L^q_{\text{loc}}(\mathbb{R}^n)) \tag{4}$$

$$\text{and} \qquad d_S^{-1} \in L^{p^*}\left(0,T; L^{q^*}_{\text{loc}}(\mathbb{R}^n)\right) \tag{5}$$

where $\frac{1}{p} + \frac{1}{p^*} = \frac{1}{q} + \frac{1}{q^*} = 1$,

then

- *for all initial data $u_0 \in L^\infty_{\text{loc}}(\mathbb{R}^n)$ there exists a unique weak solution of* (TE),
- *there exists a unique regular Lagrangian Flow solution X of* (ODE),
- *the regular Lagrangian Flow X avoids the set S, that is*

$$\mu_n(\{x \in \mathbb{R}^n \ : \ (t, X(t,x)) \in S \quad \text{for some } t \in [0,T]\}) = 0. \tag{6}$$

We prove this using the theory of renormalized solutions (after DiPerna & Lions [17] and Ambrosio [2]) which we recall in Section 1.1 in their local formulation.

The avoidance result (6) was first studied in the autonomous case by Aizenman [1], then Cipriano & Cruzeiro [8] and Robinson & Sharples [32] in the non-autonomous case, which we recall in Section 1.3. In the present work we improve these result by accounting for the direction of b: the condition (4) only requires the component of b *normal* to S to be integrable.

The condition (5) encodes some anisotropic fractal detail of the set S, first studied in Robinson & Sharples [32]. In Section 1.2 we recall the basic properties of the 'codimension print', and its relationship to the more familiar box-counting dimensions. In Section 2.3 we study the codimension prints for sets S consisting of singularities that evolve with time, obtaining the following result in terms of the box-counting dimensions of the temporal sections $S(t) := \{x \in \mathbb{R}^n \ : \ (t,x) \in S\}$ of the singular set S.

Proposition 1.4. *Suppose S_0 is compact, let $Z\colon [0,T] \times S_0 \to \mathbb{R}^n$ and suppose there exists α in the range $0 < \alpha \leq 1$ and $K > 0$ such that*

$$|Z(t_1,x) - Z(t_2,x)| \leq K|t_1 - t_2|^{\alpha} \qquad \forall t_1, t_2 \in [0,T] \quad \forall x \in S_0$$

(i.e. Z is α-Hölder continuous in t, uniformly in x).

Let $S = \{(t, Z(t,x)) \ : \ t \in [0,T], \ x \in S_0\}$. If b satisfies i), ii), iii), iv), and

$$b \cdot \nabla d_S \in L^1\left(0,T; L^q_{\mathrm{loc}}(\mathbb{R}^n)\right)$$

for some q with

$$\frac{1}{q} + \frac{1}{\alpha\left(n - \sup_{t\in[0,T]} \dim_{\mathrm{B}} S(t)\right)} < 1$$

then the conclusions of Theorem 1.3 hold. Here, $\dim_{\mathrm{B}} S(t)$ *denotes the box-counting dimension of the set $S(t)$, which is defined in Definition 1.14 below.*

1.1 Existence and Uniqueness

The existence of weak solutions of (TE) requires only the additional assumption that the (spatial) divergence $\operatorname{div} b \in L^1(0,T; L^\infty(\mathbb{R}^n))$ and follows from a standard compactness argument (see, for example, DiPerna & Lions [17] Proposition II.1). However without further regularity assumptions the uniqueness of weak solutions is not assured. Indeed, Depauw [16] constructs a divergenceless, bounded vector field b that, in addition to the trivial zero solution, admits a weak solution $u \neq 0$ with initial data $u_0 = 0$.

In their seminal paper DiPerna & Lions [17] proved that weak solutions of (TE) are unique under the additional assumptions that b has some Sobolev regularity (i.e. b is integrable and has integrable weak derivatives), and is either bounded or decays sufficiently quickly at infinity. De Lellis [14] comments that DiPerna & Lions' strategy can be decomposed into an 'easy' part and a 'hard' part. The easy part establishes uniqueness provided that every weak solution also satisfies (in a distributional sense) the 'renormalized' equation

$$\partial_t \beta(u) + b \cdot \nabla \beta(u) = 0 \qquad\qquad \text{on } (0,T) \times \mathbb{R}^n \tag{7}$$

for all maps β in an appropriate class, which trivially holds if u is a smooth solution. Formally, by setting $\beta(z) = z^2$, integrating (7), and applying the divergence theorem we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} u^2(t,x)\,\mathrm{d}x \leq \|\operatorname{div} b(t)\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} u^2(t,x)\,\mathrm{d}x, \tag{8}$$

from which we conclude with Gronwall's inequality that $u \equiv 0$ for initial data $u_0 = 0$. The uniqueness of solutions then follows from the linearity of (7). DiPerna & Lions' results make this argument rigorous for weak solutions by employing a distributional form of Gronwall's inequality (Theorem II.2 of DiPerna & Lions [17]).

The hard part of the DiPerna-Lions theory is to demonstrate that all weak solutions satisfy the renormalized equation (7) provided that b has Sobolev regularity $b \in L^1(0,T;W^{1,1}_{\text{loc}}(\mathbb{R}^n))$. Essentially, this follows from noting that for each weak solution u, its mollification $u_\varepsilon = u * \rho_\varepsilon$ satisfies $\partial_t u_\varepsilon + b \cdot \nabla u_\varepsilon = r_\varepsilon$ where ρ_ε is the standard (spatial) mollifier and $r_\varepsilon = b \cdot \nabla u_\varepsilon - (b \cdot \nabla u) * \rho_\varepsilon$ is the commutator of the second term of (TE) with respect to the mollifier. As the mollified solution is smooth, it follows that $\partial_t \beta(u_\varepsilon) + b \cdot \nabla \beta(u_\varepsilon) = \beta'(u_\varepsilon) r_\varepsilon$ for arbitrary $\beta \in C^1(\mathbb{R})$. Passing to the limit as $\varepsilon \to 0$ we conclude that u is a renormalized solution provided that the commutator $r_\varepsilon \to 0$ in $L^1_{\text{loc}}(\mathbb{R}^n)$. The technical portion of DiPerna & Lions' uniqueness result is precisely that the commutator r_ε converges for Sobolev vector fields b (Lemma II.1 of DiPerna & Lions [17]).

The DiPerna-Lions theory has been adapted to a wider class of vector fields including those that are 'piecewise Sobolev' (Lions [22]), of Vlasov type (Bouchut [5]), or have 'conormal BV' regularity (Colombini & Lerner [9]). In these cases, without Sobolev regularity of the vector field, the convergence of the commutator r_ε is highly sensitive to the choice of mollifier ρ_ε (see Capuzzo Dolcetta & Perthame[7]). These results rely on an anisotropic smoothing argument, in which the mollifier ρ_ε is locally chosen to account for the particular structure of the vector field (see [2]).

In a significant breakthrough, Ambrosio [2] extended DiPerna & Lions' theory to the large class of vector fields of bounded variation $b \in L^1(0,T;BV_{\text{loc}}(\mathbb{R}^n))$ (i.e. the spatial distributional derivative of b is a measure with finite total variation), which includes the classes considered in [22], [5], and [9]. Ambrosio's highly technical analysis uses Alberti's rank one theorem, a deep measure theoretic result, to show that at small scales any BV vector field behaves like one of the 'conormal' BV fields considered in [9].

In the present work we consider a class of vector fields that are locally in $L^1(0,T;BV_{\text{loc}}(\mathbb{R}^n))$ *except* on a set $S \subset [0,T] \times \mathbb{R}^n$ (we make this precise later). Ultimately we will require that the 'singular set' S will have a small anisotropic fractal dimension in the sense of Robinson & Sharples [32], which is related to the familiar box-counting dimensions.

Vector fields of this type appear quite naturally in point vortex models of fluid dynamics. For example, Crippa et al. [13] consider the vortex-wave system: in this 2-dimensional setting the Biot-Savart law is used to recover the velocity field b of a fluid from its vorticity $\omega = \operatorname{curl} b$, which includes an initial dirac mass at $z_0 \in \mathbb{R}^2$ that evolves along a Lipschitz trajectory $t \mapsto z(t)$. The resulting velocity field $b(t,x) = v(t,x) + (x - z(t))^\perp / |x - z(t)|^2$, with v bounded and enjoying spatial Sobolev regularity, does not have bounded variation (nor finite L^2 norm) in any neighbourhood of the trajectory of the dirac mass $S = \{(t, z(t)) \ : \ t \in [0,T]\}$, therefore falling outside the scope of Ambrosio's uniqueness result. Nevertheless, exploiting the explicit form of the singular part of b, it is proved in [13] that there exists a unique regular Lagrangian flow and that generically, its trajectories do not intersect the trajectory of the point vortex. Hence Theorem 1.3 can be seen as an extension of that result for more general singular sets. In Subsection 2.4 we will explain more in detail the link between the present work and [13]. In particular, the avoidance result of Theorem 1.3 may be simply adapted to retrieve the result of Crippa et al. [13] in a straightforward way. Similar questions also occur in the setting of Vlasov-Poisson equations with densities including point charges that generate point singularities in the electric field. We refer to the articles by Caprino et al. [6] for the two-dimensional case and by Crippa, Ligabue, Saffirio [12] for the three-dimensional case. Finally, in connection with the study of singular velocities generated by singular measure-valued vorticities in fluid dynamics, we mention the recent work by Fefferman, Pooley and Rodrigo [20] on the construction of velocity fields for active scalar systems with measure-valued solutions whose support does not satisfy conservation of the Hausdorff dimension.

1.1.1 Local renormalisation

We now recall the local formulation of the renormalization theory (see DiPerna & Lions [17], Ambrosio [2] and De Lellis [14]).

Definition 1.5. *A weak solution $u \in L^\infty_{\mathrm{loc}}((0,T)\times\mathbb{R}^n)$ of* (TE) *with initial data $u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be* renormalized *if for all $\beta \in C^1(\mathbb{R})$ the map $\beta(u)$ is a weak solution of* (TE) *with initial data $\beta(u_0)$.*

We say that b has the *renormalization property* if every weak solution of (TE) is a renormalized solution.

The formal uniqueness argument in the previous section holds for renormalized solutions:

Theorem 1.6 ([17] Theorem II.2)**.** *If the vector field b satisfies i), ii) and iii) and b has the renormalization property then for all $u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ there exists a unique solution to* (TE)*.*

To prove our main result we proceed locally: we will show that b has the renormalization property 'away from S' where it has bounded variation, and also in a neighbourhood of S, as the set is sufficiently small and b sufficiently integrable.

To define renormalization on an open set $\Omega \subset [0,T]\times\mathbb{R}^n$ we must suppress the requirement for the initial condition $\beta(u(0,\cdot)) = \beta(u_0)$ to be satisfied (as indeed $t = 0$ may not intersect Ω). The transformed initial condition, which is necessary for the Gronwall argument of Theorem 1.6, can then be recovered by extending b to negative time.

Definition 1.7. *A weak solution $u \in L^\infty_{\mathrm{loc}}((0,T)\times\mathbb{R}^n)$ of* (TE) *is said to be* locally renormalized *on an open subset $\Omega \subset (0,T)\times\mathbb{R}^n$ if for all $\beta \in C^1(\mathbb{R})$*

$$\iint_\Omega \beta(u)\left(\frac{\partial\phi}{\partial t} + b\cdot\nabla\phi + \operatorname{div} b\,\phi\right)\mathrm{d}x\,\mathrm{d}t = 0 \tag{9}$$

for all test maps $\phi \in C^\infty_c(\Omega)$.

Again, we say that b has the *local renormalization property* on Ω if every weak solution of (TE) is locally renormalized on Ω.

Remark 1.8. *If a weak solution u is locally renormalized on Ω then* (9) *in fact holds for all compactly support Lipschitz maps $\phi\colon \Omega \to \mathbb{R}$. This follows by approximating ϕ by $\phi_\varepsilon = \phi * \rho_\varepsilon$ where ρ_ε is the standard mollifier (see, for example, §4.2 of Evans & Gariepy [18]). The derivative of the mollified ϕ*

$$\nabla \phi_\varepsilon = \nabla \left(\phi * \rho_\varepsilon\right) = \left(\nabla \phi\right)_\varepsilon \to \nabla \phi$$

pointwise as $\varepsilon \to 0$ wherever the derivative $\nabla\phi$ exists, which is almost everywhere by Rademacher's Theorem. Further, as ϕ is Lipschitz the derivative $\nabla\phi$ is bounded so $\left\|\left(\nabla\phi\right)_\varepsilon\right\|_{L^\infty(\Omega_\varepsilon)} \leq \left\|\nabla\phi\right\|_{L^\infty(\Omega)}$. Consequently,

$$\left|\beta\left(u\right) b \cdot \nabla\phi_\varepsilon\right| \leq \left|\beta\left(u\right) b \cdot \nabla\phi\right| \in L^1\left(\Omega\right)$$

so by the Lebesgue Dominated Convergence Theorem $\beta\left(u\right) b\cdot\nabla\phi_\varepsilon \to \beta\left(u\right) b\cdot\nabla\phi$ in $L^1\left(\Omega\right)$. A similar treatment for the remaining terms in (9) *gives*

$$\begin{aligned} 0 = \iint_\Omega \beta\left(u\right)\left(\frac{\partial\phi_\varepsilon}{\partial t} + b\cdot\nabla\phi_\varepsilon + \operatorname{div} b\, \phi_\varepsilon\right) \mathrm{d}x\, \mathrm{d}t& \\ \to \iint_\Omega \beta\left(u\right)\left(\frac{\partial\phi}{\partial t} + b\cdot\nabla\phi + \operatorname{div} b\, \phi\right) \mathrm{d}x\, \mathrm{d}t& \end{aligned}$$

where the first equality holds as $\phi_\varepsilon \in C_c^\infty\left(\Omega\right)$ and u is a locally renormalised solution of (TE) *on Ω. Consequently* (9) *holds for compactly supported Lipschitz ϕ.*

The non-uniqueness example of Depauw [16] illustrates that local renormalization on $(0,T)\times\mathbb{R}^n$ is not sufficient for renormalization: the author constructs a vector field b and a solution u with initial data u_0 such that $\beta\left(u\right)$ is a weak solution (hence u is locally renormalized on $(0,T)\times\mathbb{R}^n$) with initial data distinct from $\beta\left(u_0\right)$ (hence u is not renormalized).

We can recover renormalization from local renormalization by using the following 'trick' made explicit in De Lellis.

Lemma 1.9 (De Lellis [15] §2.3)**.** *Let $b \in L^1_{\mathrm{loc}}\left((0,T)\times\mathbb{R}^n\right)$ and extend b to negative time with $b\left(t,\cdot\right) \equiv 0$ for $t < 0$.*

If b is locally renormalized on $(-\infty,T)\times\mathbb{R}^n$, then b is renormalized.

Finally, we give a local statement of Ambrosio's renormalisation result for vector fields of bounded variation (Ambrosio's proof proceeds locally, although the statement he gives is global).

Theorem 1.10 (Ambrosio [2] Theorem 3.5)**.** *Let $b \in L^1_{\mathrm{loc}}\left((0,T)\times\mathbb{R}^n\right)$ and extend b to negative time with $b\left(t,\cdot\right) \equiv 0$ for $t < 0$. Let $\Omega \subset (-\infty,T)\times\mathbb{R}^n$ be an open set. If*

$$\begin{aligned} b|_\Omega &\in L^1_{\mathrm{loc}}\left(-\infty,T; BV_{\mathrm{loc}}\left(\mathbb{R}^n\right)\right) \\ \textit{and} \qquad \operatorname{div} b &\in L^1_{\mathrm{loc}}\left((-\infty,T)\times\mathbb{R}^n\right) \end{aligned}$$

then b has the local renormalization property on Ω.

1.2 Fractal geometry

With many evolutionary differential equations, including the transport equation (TE), it is natural to distinguish between spatial and temporal regularity. This manifests in the Bochner spaces on which the vector field b is defined.

We will make a similar distinction between the spatial and temporal detail of the set of non-BV singularities $S \subset [0,T] \times \mathbb{R}^n$ using some tools of fractal geometry. In particular we will use the codimension print of Robinson & Sharples [32], which we recall below. This will be particularly useful for singular sets composed of trajectories (such as moving point vortices), which is the content of Theorem 2.4.

The familiar Hausdorff and box-counting dimensions (recalled below) fail to encode any anisotropic (i.e. directionally dependent) detail of a set: for example if C is the Cantor 'middle half' set, which has Hausdorff and box-counting dimensions equal to $\frac{1}{2}$, then the product set $C \times C \subset \mathbb{R}^2$ has Hausdorff and box-counting dimensions equal to 1 (see Example 7.6 in [19]). Consequently, these standard notions of fractal dimension are unable to distinguish between the product set $C \times C$ and a line segment.

The anisotropic fractal detail of subsets was first considered by Rogers [33] who adapts the Hausdorff dimension by considering a family of Hausdorff measures $\mathcal{H}^\alpha$ on $\mathbb{R}^n$ parameterised by $\alpha \in \mathbb{R}^n_+$, rather than the usual 1-parameter family of Hausdorff measures. Rogers then encodes the detail of a subset $A \subset \mathbb{R}^n$ in a 'Hausdorff dimension print', defined as the set of $\alpha \in \mathbb{R}^n_+$ such that $\mathcal{H}^\alpha(A) > 0$.

The codimension print similarly encodes the anisotropic detail of S by considering the integrability of d_S^{-1}, the reciprocal of the distance function.

Definition 1.11. *For a subset $S \subset [0,T] \times \mathbb{R}^n$ the codimension print of S is the subset*

$$\operatorname{print}(S) := \left\{ (\alpha, \beta) \in (0,\infty]^2 \quad : \quad d_S^{-1} \in L^\beta\left(0,T; L^\alpha_{\mathrm{loc}}(\mathbb{R}^n)\right) \right\}.$$

Remark 1.12. *In the previous definition, α and β may belong to the interval $(0,\infty]$. Our main result entails that the exponents belong to $[1,\infty]$, see Theorem 2.1, so we will throughout use values of (α,β) in the range $[1,\infty]$.*

Immediately we see that $\operatorname{print}(S)$ is empty if the $n+1$ dimensional Lebesgue measure $\mu_{n+1}(S) > 0$. Other basic properties of the codimension print are as follows:

Lemma 1.13 (Robinson & Sharples [32] Lemma 3.1)**.** *The codimension print reverses inclusions and is invariant under closure of sets, that is for bounded sets $S, S_1, S_2 \subset [0,T] \times \mathbb{R}^n$*

$$\begin{aligned} & S_1 \subset S_2 \Rightarrow \operatorname{print}(S_2) \subset \operatorname{print}(S_1) \\ \textit{and} \qquad & \operatorname{print}(\operatorname{cl}(S)) = \operatorname{print}(S). \end{aligned}$$

The reversal of inclusions property justifies the use of the term 'codimension' as this property is shared by the more familiar codimensions $n - \dim A$ for $A \subset \mathbb{R}^n$, which appears in the 'Minkowski Sausage' formulation of the box-counting dimension (see below).

Computing the codimension print of even elementary sets can be quite involved (see Robinson & Sharples [32] Example 3.5 for the codimension print of a singleton set). However, a portion of the codimension print can be recovered from the more elementary box-counting dimensions of the set, together with its projections.

Definition 1.14. *The upper and lower box-counting dimensions of a bounded set $A \subset \mathbb{R}^n$ are given by*

$$\dim_{\mathrm{B}} A := \limsup_{\varepsilon \to 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon}$$
$$\dim_{\mathrm{LB}} A := \liminf_{\varepsilon \to 0} \frac{\log N(A, \varepsilon)}{-\log \varepsilon}$$

respectively, where $N(A, \varepsilon)$ is the smallest number of sets with diameter at most ε that form a cover of A, or one of many similar quantities which give an equivalent definition (discussed in Falconer [19] §3.1 'Equivalent Definitions').

Another useful formulation is given in terms of the Lebesgue measure of the ε-neighbourhoods of A

$$\{d_A < \varepsilon\} := \{x \in \mathbb{R}^n \ : \ d_A(x) < \varepsilon\}.$$

Lemma 1.15 ('Minkowski Sausage' formulation)**.** *The upper and lower box-counting dimensions of a bounded set $A \subset \mathbb{R}^n$ are given by*

$$\dim_{\mathrm{B}} A = n - \liminf_{\varepsilon \to 0} \frac{\log \mu_n \left(\{d_A < \varepsilon\}\right)}{\log \varepsilon}$$
$$\dim_{\mathrm{LB}} A = n - \limsup_{\varepsilon \to 0} \frac{\log \mu_n \left(\{d_A < \varepsilon\}\right)}{\log \varepsilon}$$

Proof. See Falconer [19] §3.1 'Equivalent Definitions'. ☐

The box-counting dimension of S gives some of the 'isotropic' component of $\operatorname{print}(S)$:

Theorem 1.16. *For a bounded subset $S \subset [0, T] \times \mathbb{R}^n$*

$$\begin{aligned} \alpha &< n + 1 - \dim_{\mathrm{B}} S && \Rightarrow && (\alpha, \alpha) \in \operatorname{print}(S) \\ \alpha &> n + 1 - \dim_{\mathrm{LB}} S && \Rightarrow && (\alpha, \alpha) \notin \operatorname{print}(S) \end{aligned}$$

Proof. Follows from Remark 1 of [1]. ☐

For product sets we can get some of the 'anisotropic' component of $\operatorname{print}(S)$ from the box-counting dimensions of the component sets:

Theorem 1.17 (Robinson & Sharples [32] Theorem 3.4)**.** *For bounded subsets $\mathcal{T} \subset [0, T]$ and $A \subset \mathbb{R}^n$ the point $(\alpha, \beta) \in \operatorname{print}(\mathcal{T} \times A)$ if one of the following conditions holds:*

- $\alpha < n - \dim_{\mathrm{B}} A$,
- $\beta < 1 - \dim_{\mathrm{B}} \mathcal{T}$,
- $\alpha\beta < \alpha\left(1 - \dim_{\mathrm{B}} \mathcal{T}\right) + \beta\left(n - \dim_{\mathrm{B}} A\right)$.

Further, the point $(\alpha, \beta) \notin \operatorname{print}(\mathcal{T} \times A)$ *if*

- $\alpha\beta > \alpha\,(1 - \dim_{\mathrm{LB}} \mathcal{T}) + \beta\,(n - \dim_{\mathrm{LB}} A)$.

We remark that this theorem does not completely supersede that of Theorem 1.16: for the line $\alpha = \beta$ it follows from Theorem 1.7 that

$$\begin{aligned} \alpha < 1 + n - (\dim_{\mathrm{B}} \mathcal{T} + \dim_{\mathrm{B}} A) \quad &\Rightarrow \quad (\alpha, \alpha) \in \operatorname{print}(S) \\ \alpha > 1 + n - (\dim_{\mathrm{LB}} \mathcal{T} + \dim_{\mathrm{LB}} A) \quad &\Rightarrow \quad (\alpha, \alpha) \notin \operatorname{print}(S) \end{aligned}$$

which is weaker than Theorem 1.16 as the box-counting product inequalities

$$\dim_{\mathrm{LB}} \mathcal{T} + \dim_{\mathrm{LB}} A \leq \dim_{\mathrm{LB}} (\mathcal{T} \times A) \leq \dim_{\mathrm{B}} (\mathcal{T} \times A) \leq \dim_{\mathrm{B}} \mathcal{T} + \dim_{\mathrm{B}} A$$

can be strict (see, Robinson & Sharples [31] and Example 3.6 of [32]).

Finally, we interpret this product set result in terms of the projections of the set S.

Corollary 1.18. *For a bounded subset* $S \subset [0, T] \times \mathbb{R}^n$ *the point* $(\alpha, \beta) \in \operatorname{print}(S)$ *if one of the following holds:*

- $\alpha < n - \dim_{\mathrm{B}} P_x(S)$,
- $\beta < 1 - \dim_{\mathrm{B}} P_t(S)$,
- $\alpha\beta < \alpha\,(1 - \dim_{\mathrm{B}} P_t(S)) + \beta\,(n - \dim_{\mathrm{B}} P_x(S))$.

where $P_t(S)$ *and* $P_x(S)$ *are the temporal and spatial projections of* S *respectively.*

Proof. Follows from the inclusion $S \subset P_t(S) \times P_x(S)$, Lemma 1.13 and Theorem 1.17. □

1.3 Avoidance

In the classical framework for ODEs, Aizenman [1] considered vector fields b that are smooth (or Lipschitz) on the complement of some singular set.

In general there is no flow solution of (ODE) in this setting as typically some trajectories will intersect S. However if almost every trajectory does not intersect (the closure of) S then, as the trajectories are unique and defined for all time, this aggregate of trajectories gives a unique flow solution defined almost everywhere.

To formalise this argument, Aizenman considered an aggregate of 'local' trajectories together with their existence times, and provided conditions for almost every 'local' trajectory to avoid the set S.

As we will obtain existence of solutions using the renormalisation methods, we can instead for convenience define avoidance of sets in terms of an existing regular Lagrangian flow:

Definition 1.19. *A regular Lagrangian flow* $X\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ *avoids a closed set* $S \subset [0, T] \times \mathbb{R}^n$ *if*

$$\mu_n\left(\{x \in \mathbb{R}^n \ : \ (t, X\,(t, x)) \in S \textit{ for some } t \in [0, T]\}\right) = 0.$$

In these terms, Aizenman proved the following 'autonomous' avoidance result:

Theorem 1.20 (Aizenman [1])**.** *Let $b \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$. A regular Lagrangian flow solution $X\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ of* (ODE) *avoids a set $[0,T] \times A$ if*

$$\frac{1}{q} + \frac{1}{n - \dim_{\mathrm{B}} A} < 1.$$

In Robinson & Sharples [32] this result was adapted to the non-autonomous setting:

Theorem 1.21 (Robinson & Sharples [32])**.** *A regular Lagrangian flow solution $X\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ of* (ODE) *avoids a set $S \subset [0,T] \times \mathbb{R}^n$ if*

$$b \in L^p\left(0,T; L^q_{\mathrm{loc}}(\mathbb{R}^n)\right) \tag{10}$$

and

$$d_S^{-1} \in L^{p^*}\left(0,T; L^{q^*}(\mathbb{R}^n)\right)$$

(i.e. $(q^*,p^*) \in \mathrm{print}(S)$) where $\frac{1}{p} + \frac{1}{p^*} = \frac{1}{q} + \frac{1}{q^*} = 1$.

In the present work we improve this result to account for the direction of the vector field near the set S. This is appropriate for the analysis of point-vortices as the Biot-Savart law generates a vector field perpendicular to the singular set.

In Section 2 we prove the following.

Theorem 1.22. *A regular Lagrangian flow solution $X\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ of* (ODE) *avoids a set $S \subset [0,T] \times \mathbb{R}^n$ if*

$$b \cdot \nabla d_S \in L^p\left(0,T; L^q_{\mathrm{loc}}(\mathbb{R}^n)\right),$$

and

$$d_S^{-1} \in L^{p^*}\left(0,T; L^{q^*}(\mathbb{R}^n)\right)$$

(i.e. $(q^*,p^*) \in \mathrm{print}(S)$) where $\frac{1}{p} + \frac{1}{p^*} = \frac{1}{q} + \frac{1}{q^*} = 1$.

This result relies on the validity of the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} d_S\left(t, X(t,x)\right) = \frac{\partial}{\partial t} d_S\left(t, X(t,x)\right) + \frac{\partial}{\partial t} X(t,x) \cdot \nabla d_S\left(t, X(t,x)\right) \tag{11}$$

which is not immediate as d_S is only Lipschitz continuous and X is only absolutely continuous in t for almost every $x \in \mathbb{R}^n$.

Avoidance results can yield interesting qualitative properties of a regular Lagrangian flow: Robinson & Sadowski [29] demonstrate the almost everywhere uniqueness of particle trajectories for suitable weak solutions of the Navier-Stokes equations using avoidance methods (See also [28] and [30]).

2 Proofs of the main results

2.1 Existence and Uniqueness of Solutions

We start by proving the first part of Theorem 1.3, which may be formulated as follows:

Theorem 2.1. *Let $S \subset [0,T] \times \mathbb{R}^n$ be compact. If the vector field b satisfies*

i) $b \in L^1_{\mathrm{loc}}((0,T) \times \mathbb{R}^n)$,

ii) $\operatorname{div} b \in L^1(0,T; L^\infty(\mathbb{R}^n))$,

iii) $\frac{b}{1+|x|} \in L^1((0,T) \times \mathbb{R}^n) + L^1(0,T; L^\infty(\mathbb{R}^n))$,

iv) *for all* $\Omega \subset\subset S^c$*, the vector field* b *extended by* $b(t,\cdot) \equiv 0$ *for* $t < 0$ *has the renormalization property on* Ω*.*

v) *for some* $1 \leq p, q \leq \infty$

$$b \cdot \nabla d_S \in L^p(0,T; L^q_{\mathrm{loc}}(\mathbb{R}^n)) \quad \text{and} \qquad d_S^{-1} \in L^{p^*}\left(0,T; L^{q^*}_{\mathrm{loc}}(\mathbb{R}^n)\right) \tag{12}$$

(i.e. $(q^*, p^*) \in \operatorname{print}(S)$*) where* $\frac{1}{p} + \frac{1}{p^*} = \frac{1}{q} + \frac{1}{q^*} = 1$,

then for all initial data $u_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ *there exists a unique weak solution of* (TE).

Corollary 2.2. *With the same hypotheses, there exists a unique regular Lagrangian flow solution of* (ODE).

Proof. From the DiPerna-Lions theory Theorem 1.6 it is sufficient to demonstrate that the vector field b has the renormalization property.

Let $u \in L^\infty((-\infty,T) \times \mathbb{R}^n)$ be a weak solution of (TE) on $(-\infty,T) \times \mathbb{R}^n$ where the vector field b is extended by zero for negative time. Let $\beta \in C^1(\mathbb{R})$ and $\phi \in C_c^\infty((-\infty,T) \times \mathbb{R}^n)$. Fix ε in the range $0 < \varepsilon < 1$ and define

$$\phi_\varepsilon(t,x) := \phi(t,x)\, \chi_0(d_S(t,x)/\varepsilon) \tag{13}$$

where $\chi_0 \in C^\infty(\mathbb{R})$ satisfies

$$\chi_0(z) = \begin{cases} 0 & |z| \leq \frac{1}{2} \\ 1 & |z| \geq 1. \end{cases}$$

Observe that ϕ_ε is Lipschitz (although not necessarily smooth). An unpublished result of Serrin [35] (see also [21], [36], [4], and [25]) ensures that the chain rule applies almost everywhere for the composition of the Lipschitz functions χ_0 and d_S, hence

$$\partial_t \phi_\varepsilon = \phi \chi_0'(d_S/\varepsilon) \frac{1}{\varepsilon} \partial_t d_S + \chi_0(d_S/\varepsilon)\, \partial_t \phi \tag{14}$$

and

$$\nabla \phi_\varepsilon = \phi \chi_0'(d_S/\varepsilon) \frac{1}{\varepsilon} \nabla d_S + \chi_0(d_S/\varepsilon)\, \nabla \phi \tag{15}$$

almost everywhere on $(-\infty,T) \times \mathbb{R}^n$.

For brevity we adopt the notation

$$\{d_S > \varepsilon\} := \{(t,x) \in (-\infty,T) \times \mathbb{R}^n \ : \ d_S(t,x) > \varepsilon\},$$

and similarly define the sets $\{d_S \geq \varepsilon\}$ and $\{d_S < \varepsilon\}$.

Let

$$I(\varepsilon) := \iint\limits_{\{d_S > \varepsilon\}} \beta(u) \left(\frac{\partial \phi}{\partial t} + b \cdot \nabla \phi + \operatorname{div} b \, \phi \right) \, \mathrm{d}x \, \mathrm{d}t$$

so, as $\mu_{n+1}(S) = 0$,

$$\lim_{\varepsilon \to 0} I(\varepsilon) = \int\limits_{-\infty}^{T} \int_{\mathbb{R}^n} \beta(u) \left(\frac{\partial \phi}{\partial t} + b \cdot \nabla \phi + \operatorname{div} b \, \phi \right) \, \mathrm{d}x \, \mathrm{d}t.$$

Now, as $\phi(t,x) = \phi_\varepsilon(t,x)$ on $\{d_S > \varepsilon\}$ it follows that

$$\begin{aligned} I(\varepsilon) &= \iint\limits_{\{d_S > \varepsilon\}} \beta(u) \left(\frac{\partial \phi_\varepsilon}{\partial t} + b \cdot \nabla \phi_\varepsilon + \operatorname{div} b \, \phi_\varepsilon \right) \, \mathrm{d}x \, \mathrm{d}t \\ &= \iint\limits_{\{d_S > \varepsilon/4\}} \beta(u) \left(\frac{\partial \phi_\varepsilon}{\partial t} + b \cdot \nabla \phi_\varepsilon + \operatorname{div} b \, \phi_\varepsilon \right) \, \mathrm{d}x \, \mathrm{d}t \qquad (16) \\ &\quad - \iint\limits_{\{\varepsilon \geq d_S > \varepsilon/4\}} \beta(u) \left(\frac{\partial \phi_\varepsilon}{\partial t} + b \cdot \nabla \phi_\varepsilon + \operatorname{div} b \, \phi_\varepsilon \right) \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

By assumption *iv)* and Theorem 1.9 the vector field b has the local renormalization property on $\Omega = \{d_S > \varepsilon/4\}$. Consequently, as $\operatorname{supp} \phi_\varepsilon \subset\subset \{d_S > \varepsilon/4\}$ and ϕ_ε is Lipschitz it follows by Remark 1.8 that that the integral (16) vanishes.

$$\begin{aligned} I(\varepsilon) &= - \iint\limits_{\{\varepsilon \geq d_S > \varepsilon/4\}} \beta(u) \left(\frac{\partial \phi_\varepsilon}{\partial t} + b \cdot \nabla \phi_\varepsilon + \operatorname{div} b \, \phi_\varepsilon \right) \, \mathrm{d}x \, \mathrm{d}t \\ &= - \iint\limits_{\{d_S \leq \varepsilon\}} \beta(u) \left(\frac{\partial \phi_\varepsilon}{\partial t} + b \cdot \nabla \phi_\varepsilon + \operatorname{div} b \, \phi_\varepsilon \right) \, \mathrm{d}x \, \mathrm{d}t \qquad (17) \end{aligned}$$

as ϕ_ε vanishes on $\{d_S \leq \varepsilon/4\}$.

It remains to demonstrate that (17) vanishes at the limit: as $|\phi_\varepsilon| \leq |\phi|$ and $\operatorname{div} b \in L^1((0,T) \times \mathbb{R}^n)$ we immediately obtain

$$\lim_{\varepsilon \to 0} \iint\limits_{\{d_S \leq \varepsilon\}} \beta(u) \operatorname{div} b \, \phi_\varepsilon \, \mathrm{d}x \, \mathrm{d}t = 0$$

so it is sufficient to demonstrate that the limit of

$$J(\varepsilon) := \iint\limits_{\{d_S \leq \varepsilon\}} \left| \beta(u) \left(\frac{\partial \phi_\varepsilon}{\partial t} + b \cdot \nabla \phi_\varepsilon \right) \right| \, \mathrm{d}x \, \mathrm{d}t$$

is zero.

Using the identities (14) and (15),

$$
\begin{aligned}
J(\varepsilon) = \iint_{\{d_S \le \varepsilon\}} \Big| &\beta(u)\,\phi\chi_0'(d_S/\varepsilon)\,\frac{1}{\varepsilon}\partial_t d_S + \beta(u)\,\chi_0(d_S/\varepsilon)\,\partial_t\phi \\
&+ \beta(u)\,\phi\chi_0'(d_S/\varepsilon)\,\frac{1}{\varepsilon} b\cdot\nabla d_S + \beta(u)\,\chi_0(d_S/\varepsilon)\,b\cdot\nabla\phi\Big| \,\mathrm{d}x\,\mathrm{d}t. \\
\le \iint_{\{d_S\le\varepsilon\}} &\frac{1}{\varepsilon}\left|\beta(u)\,\phi\chi_0'(d_S/\varepsilon)\,(\partial_t d_S + b\cdot\nabla d_S)\right| \,\mathrm{d}x\,\mathrm{d}t \qquad (18)\\
+ \iint_{\{d_S\le\varepsilon\}} &\left|\beta(u)\,\chi_0(d_S/\varepsilon)\,(\partial_t\phi + b\cdot\nabla\phi)\right| \,\mathrm{d}x\,\mathrm{d}t. \qquad (19)
\end{aligned}
$$

As χ_0 and $\beta(u)$ are bounded, and $b \in L^1_{\text{loc}}((0,1)\times\mathbb{R}^n)$ the integral (19) vanishes as $\varepsilon \to 0$. Writing $J_1(\varepsilon)$ for the integral (18) we see that

$$
J_1(\varepsilon) \le C\frac{1}{\varepsilon}\iint_{\{d_S\le\varepsilon\}} 1 + |b\cdot\nabla d_S| \,\mathrm{d}x\,\mathrm{d}t
$$

as $\beta(u)$, ϕ and χ_0' are bounded, and $|\partial_t d_S| \le 1$ as distance functions have Lipschitz constant 1. Applying Hölder's inequality with exponents satisfying $v)$ we obtain

$$
J_1(\varepsilon) \le C\frac{1}{\varepsilon}\left\|(1+|b\cdot\nabla d_S|)\,|_{\{d_S\le\varepsilon\}}\right\|_{L^p(-1,T;L^q(\mathbb{R}^n))}\left(\int_{-1}^{T}\left(\int_{\{x\in\mathbb{R}^n\ :\ d_S(t,x)\le\varepsilon\}} 1\,\mathrm{d}x\right)^{\frac{p^*}{q^*}}\mathrm{d}t\right)^{\frac{1}{p^*}}.
$$

Now, by Chebyshev's inequality

$$
\begin{aligned}
&\frac{1}{\varepsilon}\left(\int_{-1}^{T}\mu_n\left(\left\{x\in\mathbb{R}^n\ :\ d_S(t,x)^{-1} > 1/\varepsilon\right\}\right)^{\frac{p^*}{q^*}}\mathrm{d}t\right)^{\frac{1}{p^*}} \qquad (20)\\
&\le \frac{1}{\varepsilon}\left(\int_{-1}^{T}\left(\varepsilon^{q^*}\int\limits_{\{x\in\mathbb{R}^n\ :\ d_S(t,x)\le\varepsilon\}} d_S(t,x)^{-q^*}\,\mathrm{d}x\right)^{\frac{p^*}{q^*}}\mathrm{d}t\right)^{\frac{1}{p^*}}\\
&= \left\|d_S^{-1}|_{\{d_S\le\varepsilon\}}\right\|_{L^{p^*}(-1,T;L^{q^*}(\mathbb{R}^n))}.
\end{aligned}
$$

Hence

$$
J_1(\varepsilon) \le C\left\|(1+|b\cdot\nabla d_S|)\,|_{\{d_S\le\varepsilon\}}\right\|_{L^p(-1,T;L^q(\mathbb{R}^n))}\left\|d_S^{-1}|_{\{d_S\le\varepsilon\}}\right\|_{L^{p^*}(-1,T;L^{q^*}(\mathbb{R}^n))}.
$$

This tends to zero as $\varepsilon\to 0$ as $1+|b\cdot\nabla d_S| \in L^p(-1,T;L^q_{\text{loc}}(\mathbb{R}^n))$ from (4) and $d_S^{-1}\in L^{p^*}\left(-1,T;L^{q^*}_{\text{loc}}(\mathbb{R}^n)\right)$ from (5). Consequently,

$$
\lim_{\varepsilon\to 0} I(\varepsilon) = \int\limits_{-\infty}^{T}\int_{\mathbb{R}^n}\beta(u)\left(\frac{\partial\phi}{\partial t} + b\cdot\nabla\phi + \operatorname{div} b\,\phi\right)\mathrm{d}x\,\mathrm{d}t = 0
$$

As $\beta\in C^1(\mathbb{R})$ was an arbitrary map and $u\in L^\infty((0,T)\times\mathbb{R}^n)$ an arbitrary weak solution it follows that the vector field b has the renormalization property on $(-\infty,T)\times\mathbb{R}^n$. $\square$

Remarks:

- From (20) it is sufficient for the spatial component of d_S^{-1} to be locally weak-$L^{q^*}(\mathbb{R}^n)$.
- It is straightforward (but notationally demanding) to adapt the above proof for unbounded weak solutions $u \in L^p(0,T;L^q(\mathbb{R}^n))$.

2.2 Avoidance of Singularities

We now show that almost every trajectory of the regular Lagrangian Flow does not intersect the set S, namely the second part of Theorem 1.3.

Theorem 2.3. *Let $S \subset [0,T] \times \mathbb{R}^n$ be compact, and suppose that the assumption v) of Theorem 2.1 is satisfied. If X is a regular Lagrangian flow solution of* (ODE) *then X avoids the set S.*

Proof. Let $\Omega = \{x \in \mathbb{R}^n \ : \ X(t,x) \in S \text{ for some } t \in (0,T]\}$. For each $r_0 > 0$ and $0 < \delta < r_0$ define

$$F(\delta) = \{x \in \Omega \ : \ d_S(0,x) \geq r_0,\ \tau_\delta(x) < T\}$$

where

$$\tau_\delta(x) := \begin{cases} \sup\{t^* \ : \ d_S(t,X(t,x)) \geq \delta \quad \forall t \in [0,t^*]\} & \text{if } d_S(0,x) > \delta \\ 0 & \text{if } d_S(0,x) \leq \delta. \end{cases}$$

Following Aizenman [1] and Robinson & Sharples [32] it is sufficient to show that $\mu_n(F(\delta)) \to 0$ as $\delta \to 0$.

Define the Lipschitz function

$$g(y) = \begin{cases} \log(r_0/y) & \delta \leq y \leq r_0 \\ 0 & r_0 < y \end{cases}$$

and note that

$$g(d_S(0,x)) = 0 \qquad \forall x \in F(\delta) \tag{21}$$

$$g(d_S(\tau_\delta(x), X(\tau_\delta(x),x))) = g(\delta) \qquad \text{a.e. } x \in F(\delta) \tag{22}$$

as the trajectories $t \mapsto X(t,x)$ are continuous for almost every $x \in \mathbb{R}^n$.

Now d_S is Lipschitz with Lipschitz constant 1 so by Rademacher's Theorem there is a set N with $\mu_{n+1}(N) = 0$ such that the derivatives $\frac{\partial}{\partial t} d_S(t,x)$ and $\nabla d_S(t,x)$ exist, and are bounded by 1, for all $(t,x) \notin N$. The compressibility constant (3) then ensures that $\frac{\partial}{\partial t} d_S(t,X(t,x))$ and $\nabla d_S(t,X(t,x))$ exist, and are bounded by 1, for almost every $(t,x) \in [0,T] \times \mathbb{R}^n$.

Further, as trajectories are absolutely continuous, for almost every $x \in \mathbb{R}^n$

$$t \mapsto d_S(t,X(t,x))$$

is absolutely continuous, hence is differentiable for almost every $t \in [0,T]$. It follows from Marcus & Mizel [25] that for almost every $x \in \mathbb{R}^n$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} d_S(t,X(t,x)) &= \frac{\partial}{\partial t} d_S(t,X(t,x)) + \frac{\partial X}{\partial t} \cdot \nabla d_S(t,X(t,x)) \quad \text{a.e. } t \in [0,T] \\ &= \frac{\partial}{\partial t} d_S(t,X(t,x)) + b(t,X(t,x)) \cdot \nabla d_S(t,X(t,x)) \quad \text{a.e. } t \in [0,T] \end{aligned}$$

hence for almost every $x \in \mathbb{R}^n$, for almost every $t \in [0,T]$

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} d_S\left(t, X\left(t,x\right)\right) \right| \leq 1 + \left| b\left(t, X\left(t,x\right)\right) \cdot \nabla d_S\left(t, X\left(t,x\right)\right) \right|. \tag{23}$$

Next, as g is Lipschitz, it follows from Serrin & Varberg [34] that for almost every $x \in \mathbb{R}^n$, for almost every $t \in [0,T]$

$$\frac{\mathrm{d}}{\mathrm{d}t} g\left(d_S\left(t, X\left(t,x\right)\right)\right) = g'\left(d_S\left(t, X\left(t,x\right)\right)\right) \frac{\mathrm{d}}{\mathrm{d}t} d_S\left(t, X\left(t,x\right)\right). \tag{24}$$

Now, from (21) and (22)

$$\begin{aligned} \mu_n\left(F\left(\delta\right)\right) \left|g\left(\delta\right)\right| &= \int_{F(\delta)} \left| g\left(d_S\left(\tau_\delta(x), X\left(\tau_\delta(x), x\right)\right)\right) - g\left(d_S\left(0,x\right)\right) \right| \,\mathrm{d}x \\ &= \int_{F(\delta)} \left| \int_0^{\tau_\delta(x)} \frac{\mathrm{d}}{\mathrm{d}t} g\left(d_S\left(t, X\left(t,x\right)\right)\right) \,\mathrm{d}t \right| \,\mathrm{d}x \end{aligned}$$

from (23) and (24)

$$\leq \int_{F(\delta)} \int_0^{\tau_\delta(x)} \left| g'\left(d_S\left(t, X\left(t,x\right)\right)\right) \right| \left(1 + \left| b\left(t, X\left(t,x\right)\right) \cdot \nabla d_S\left(t, X\left(t,x\right)\right) \right|\right) \,\mathrm{d}t \,\mathrm{d}x.$$

As the integrand is measurable from $[0,T] \times \mathbb{R}^n \to \mathbb{R}$ we apply Fubini's Theorem

$$\begin{aligned} &= \int_0^T \int_{\mathbb{R}^n} \left| g'\left(d_S\left(t, X\left(t,x\right)\right)\right) \right| \left(1 + \left| b\left(t, X\left(t,x\right)\right) \cdot \nabla d_S\left(t, X\left(t,x\right)\right) \right|\right) \,\mathrm{d}x \,\mathrm{d}t \\ &\leq L \int_0^T \int_{\mathbb{R}^n} \left| g'\left(d_S\left(t,x\right)\right) \right| \left(1 + \left| b\left(t,x\right) \cdot \nabla d_S\left(t,x\right) \right|\right) \,\mathrm{d}x \,\mathrm{d}t. \end{aligned}$$

Finally, as

$$g'\left(y\right) = \begin{cases} -\frac{1}{y} & \delta < y < r_0 \\ 0 & y > r_0 \end{cases}.$$

we conclude that

$$\mu_n\left(F\left(\delta\right)\right) \left|g\left(\delta\right)\right| \leq L \int_0^T \int_{\{x \,:\, d_S(t,x) < r_0\}} d_S\left(t,x\right)^{-1} \left(1 + \left| b\left(t,x\right) \cdot \nabla d_S\left(t,x\right) \right|\right) \,\mathrm{d}x \,\mathrm{d}t$$

which is finite from Hölder's inequality and the conditions (4) and (v)). As $\left|g\left(\delta\right)\right| = \log\left(r_0/\delta\right) \to 0$ as $\delta \to 0$, it follows that $\mu_n\left(F\left(\delta\right)\right) \to 0$ as required. $\square$

2.3 Codimension print of trajectories

In many applications we wish to consider the anisotropic detail of the graph of trajectories

$$S := \left\{ \left(t, Z\left(t,x\right)\right) \,:\, t \in [0,T], x \in S_0 \right\} \subset [0,T] \times \mathbb{R}^n$$

where $S_0 \subset \mathbb{R}^n$ is some set of initial data. For sufficiently regular maps Z this graph will have similar anisotropic detail to the product $[0,T] \times S_0$ in which case the codimension print would be immediately given by Theorem 1.17.

If the map $(t,x) \mapsto (t, Z(t,x))$ is bi-Lipschitz then it is not difficult to show that

$$d_{[0,T]\times S_0}(t,x) \le C d_S(t, Z(t,x)) \le C^2 d_{[0,T]\times S_0}(t,x)$$

from which it follows that the codimension prints of S and $[0,T]\times S_0$ are identical, in which case the codimension print of S is immediately given by Theorem 1.17.

In general the situation is more complicated: first, if an individual trajectory is not Lipschitz in time then the graph it traces can have a large fractal dimension. Secondly, if Z is not bi-Lipschitz in space then the box-counting dimension of the temporal section $S(t)$ may vary in time.

For example if the trajectories are described by $Z(t,x) := x + t\left(x^2 - x\right)$ then the set of initial data $S(0) := \left\{n^{-1} \;:\; n \in \mathbb{N}\right\}$ evolves to the set $S(1) = \{Z(1,x) \;:\; x \in S(0)\} = \left\{n^{-2} \;:\; n \in \mathbb{N}\right\}$, in which case the upper and lower box-counting dimensions are not preserved as

$$\dim_{LB} S(0) = \dim_B S(0) = \frac{1}{2} > \frac{1}{3} = \dim_{LB} S(1) = \dim_B S(1)$$

(see Example 13.4 of Robinson [27]).

However, if the trajectories have some uniform Hölder regularity in time then we can describe the codimension print in terms of the Hölder exponents and the maximum box-counting dimension of the temporal sections. This result requires no spatial regularity of the map Z.

Theorem 2.4. *Let $S_0 \subset \mathbb{R}^n$ be bounded. Suppose that for some α in the range $0 < \alpha \le 1$ the map $Z\colon [0,T]\times S_0 \to \mathbb{R}^n$ is α-Hölder continuous in t uniformly in x, which is to say that there exists a $K > 0$ with*

$$|Z(t_1,x) - Z(t_2,x)| \le K|t_1 - t_2|^\alpha \qquad \forall t_1,t_2 \in [0,T] \quad \forall x \in S_0, \tag{25}$$

then the distance function d_S of the set

$$S := \{(t, Z(t,x)) \;:\; t \in [0,T], x \in S_0\} \subset [0,T]\times\mathbb{R}^n \tag{26}$$

satisfies

$$d_S^{-1} \in L^\infty\left(0,T; L^r_{\mathrm{loc}}(\mathbb{R}^n)\right),$$

(that is $(r,\infty) \in \operatorname{print}(S)$), for all $r < \alpha\left(n - \sup_{t\in[0,T]} \dim_B S(t)\right)$.

It is clear that $d_S(t,x) \le d_{S(t)}(x)$ as for all $y \in S(t)$ the point $(t,y) \in S$ and so $d_S(t,x) \le |(x,t) - (y,t)| = |x-y|$. In the following proof we see that the Hölder condition (25) ensures that the converse inequality $d_{S(t)}(x) \le (K+1)\, d_S(t,x)^\alpha$ holds.

Proof. From Lemma 1.13 the codimension print is invariant under closure of sets so we can assume that S is closed. Let $r < \alpha\left(n - \sup_{t\in[0,T]} \dim_B S(t)\right)$ and let $\delta > 0$ be sufficiently small that $r + \delta < \alpha\left(n - \sup_{t\in[0,T]} \dim_B S(t)\right)$.

To show that $d_S^{-1} \in L^\infty\left(0,T;L^r_{\mathrm{loc}}\left(\mathbb{R}^n\right)\right)$ it is sufficient to demonstrate that

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int\limits_{\{x\ :\ d_S(t,x)<1\}} d_S\left(t,x\right)^{-r} \mathrm{d}x \tag{27}$$

is finite, as d_S^{-1} is bounded away from S.

Fix $(t,x) \in [0,T]\times\mathbb{R}^n$ such that $d_S\left(t,x\right)<1$ and let $(s,y)\in S$ be such that $d_S\left(t,x\right) = \left|(t,x)-(s,y)\right|$. Now, $y = Z\left(s,y_0\right)$ for some $y_0 \in S_0$, as S has the form (26), so certainly the point $Z\left(t,y_0\right) \in S\left(t\right)$. Consequently,

$$d_{S(t)}\left(x\right) \le \left|x - Z\left(t,y_0\right)\right| \le \left|x - Z\left(s,y_0\right)\right| + \left|Z\left(s,y_0\right) - Z\left(t,y_0\right)\right|$$

which, from the uniform Hölder condition (25),

$$\begin{aligned} &\le \left|x-y\right| + K\left|t-s\right|^\alpha \le d_S\left(t,x\right) + K d_S\left(t,x\right)^\alpha \\ &\le \left(K+1\right) d_S\left(t,x\right)^\alpha \end{aligned} \tag{28}$$

as $d_S\left(t,x\right)<1$ and $\alpha\le1$. This inequality yields the inclusion

$$\left\{x\ :\ d_S\left(t,x\right)<1\right\} \subset \left\{x\ :\ d_{S(t)}\left(x\right) < K+1\right\} \qquad \forall t\in[0,T]$$

and the inequality $d_S\left(t,x\right)^{-1} \le \left(\frac{1}{K+1} d_{S(t)}\left(x\right)\right)^{-1/\alpha}$ so for all $t\in[0,T]$

$$I\left(t\right) := \int\limits_{\{x\ :\ d_S(t,x)<1\}} d_S\left(t,x\right)^{-r}\mathrm{d}x \le \int\limits_{\{x\ :\ d_{S(t)}(x)<K+1\}} d_{S(t)}\left(x\right)^{-r/\alpha}\mathrm{d}x. \tag{29}$$

We write $M := (K+1)$ and, following the argument of [1], we rewrite (29) as

$$\begin{aligned} I\left(t\right) &\le \int\limits_{\{x\ :\ d_{S(t)}(x)<M\}} M^{-r/\alpha}\mathrm{d}x + \int\limits_{\{x\ :\ d_{S(t)}(x)<M\}} \left(d_{S(t)}\left(x\right)^{-r/\alpha} - M^{-r/\alpha}\right)\mathrm{d}x \\ &= M^{-r/\alpha}\mu_n\left(\left\{x\ :\ d_{S(t)}\left(x\right)<M\right\}\right) + \int\limits_{\{x\ :\ d_{S(t)}(x)<M\}} \int_{M^{-r/\alpha}}^{d_{S(t)}(x)^{-r/\alpha}} 1\mathrm{d}s\,\mathrm{d}x \end{aligned}$$

which, from Fubini's theorem,

$$= M^{-r/\alpha}\mu_n\left(\left\{x\ :\ d_{S(t)}\left(x\right)<M\right\}\right) + \int_{M^{-r/\alpha}}^{\infty} \mu_n\left(\left\{x\ :\ d_{S(t)}\left(x\right) < s^{-\alpha/r}\right\}\right)\mathrm{d}s.$$

Next, the diameter of the temporal sections $\omega := \sup_{t\in[0,T]}\operatorname{diam} S\left(t\right) < \infty$ as S is bounded, so from Lemma 1.15 there exists a constant C dependent on $r/\alpha + \delta/\alpha$ and ω such that

$$\mu_n\left(\left\{d_{S(t)} < \varepsilon\right\}\right) \le C\varepsilon^{r/\alpha+\delta/\alpha} \qquad \forall\varepsilon\in(0,M] \quad \forall t\in[0,T].$$

Consequently,

$$I\left(t\right) \le M^{-r/\alpha}CM^{r/\alpha+\delta/\alpha} + \int_{M^{-r/\alpha}}^{\infty} C\left(s^{-\alpha/r}\right)^{r/\alpha+\delta/\alpha}\mathrm{d}s,$$

which is finite as $\left(\alpha/r\right)\left(r/\alpha+\delta/\alpha\right) = 1+\delta/r > 1$. Further, this bound is independent of t so we conclude that (27) is finite from which it follows that $d_S^{-1}\in L^\infty\left(0,T;L^r_{\mathrm{loc}}\left(\mathbb{R}^n\right)\right)$, as required. □

2.4 Application to the vortex-wave system

The purpose of this paragraph to explore the link between our main result stated in Theorem 1.3 and a situation arising in fluid dynamics to describe the interaction of a 2-dimensional fluid with several point vortices. The resulting system, called vortex-wave system, is a coupling of a nonlinear transport equation for the vorticity of the fluid and a system of ODE for the evolution of the point vortices. It is derived from the incompressible Euler equations and was introduced by Marchioro and Pulvirenti [24, 23] and by Starovoĭtov [37, 38]. In the case of one single point vortex, the vortex-wave system may be written as

$$\begin{cases} \partial_t \omega + b \cdot \nabla \omega = 0 \\ v = \dfrac{x^\perp}{|x|^2} * \omega, \quad b = v + \dfrac{(x - z(t))^\perp}{|x - z(t)|^2} \\ \dot{z}(t) = v(t, z(t)), \end{cases} \tag{30}$$

with $\omega \in L^\infty((0,T), L^1 \cap L^p(\mathbb{R}^2))$ for some $p > 2$, denoting the vorticity, and $z \in W^{1,\infty}((0,T))$ denoting the vortex trajectory. Note that $p > 2$ indeed ensures that the velocity field generated by ω, $v = \frac{x^\perp}{|x|^2} * \omega$, is uniformly bounded on $(0,T) \times \mathbb{R}^2$. In Crippa et al. [13], it is proved that for any such weak solution (ω, z) to the system (30), there exists a unique regular Lagrangian flow such that ω is constant along the flow trajectories. Moreover, generically, its trajectories do not collide with the trajectory of the point vortex (namely $X(t,x)$ avoids the set $S(t) = \{z(t)\}$ for almost every $x \in \mathbb{R}^2$). Actually, this property is proved in [13] for more general vector fields b of the form above, where v is a given bounded vector field satisfying assumptions that are essentially the ones required by Ambrosio's result. In particular, the point vortex trajectory $t \mapsto z(t)$ is Lipschitz.

When there is only one point vortex trajectory, the singular set is defined by

$$S = \{(t, z(t)); \quad t \in [0,T]\} \tag{31}$$

and its temporal sections are $S(t) = \{z(t)\}$, so that $\dim_B(S(t)) = 0$ for all $t \in [0,T]$, and $d_{S(t)}(x) = |x - z(t)|$.

Moreover, since z is Lipschitz (with Lipschitz constant given K), we have

$$\frac{1}{K+1} d_{S(t)}(x) \leq d_S(t,x) \leq d_{S(t)}(x) \leq (K+1) d_S(t,x). \tag{32}$$

So by Theorem 2.4 we retrieve that $d_S^{-1} \in L^\infty(0,T; L^r_{\text{loc}}(\mathbb{R}^2))$ for all $r < 2$. In order to apply Proposition 1.4 yielding Theorem 1.3, we still need to check that $b \cdot \nabla d_S \in L^1\left(0,T; L^q_{\text{loc}}\left(\mathbb{R}^2\right)\right)$ for some $q > 2$. However we do not know the explicit form of d_S in the present setting; so, we need to come back to the explicit distance $d_{S(t)}(x)$.

In [13], the renormalization property is obtained by considering test functions depending on the quantity $d_{S(t)}(x)/\varepsilon = |x - z(t)|/\varepsilon$. It is then based on the observation that $b \cdot \nabla d_{S(t)}(x) = v \cdot \nabla d_{S(t)}(x)$ is not singular. So, the first part of Theorem 1.3 (namely of Theorem 2.1), which is based on test functions defined by (13), is an extension of the method of [13] to more general singular fields and sets. Theorem 2.3 is also an avoidance property applying to more general singular sets. We remark that it can be used to retrieve easily the avoidance property established in [13]. Indeed, coming back to the proof of Theorem 2.3, we get by virtue of (32)

$$\delta \leq d_{S(\tau_\delta)}(x) \leq (K+1)\delta, \quad \forall x \in F(\delta). \tag{33}$$

We can assume that δ is sufficienly small so that $(K+1)\delta \leq \delta^{1/2}$. Hence

$$g(\delta) = \log(1/\delta) \leq 2\log(1/d_{S(\tau_\delta)}(x)) = g(d_{S(\tau_\delta)}(x)) - g(d_{S_0}(x)).$$

Hence we may mimick the subsequent computations, replacing $d_S(t, X(t,x))$ by $d_{S(t)}(X(t,x))$ and observing the cancellation on $b(t,\cdot)\cdot\nabla d_{S(t)}$. It follows that $\mu_2(F(\delta)) \to 0$ as $\delta \to 0$. Note that this argument would apply equally to a point vortex trajectory only Hölder continuous in time.

We finally mention that this applies to a finite number of point vortex trajectories that do not collide on $[0,T]$: this follows from a straightforward adaptation of the previous arguments.

3 Conclusion

We have demonstrated that the renormalization theory of DiPerna & Lions and Ambrosio can be extended to vector fields that are BV off a set of singularities, provided that the anisotropic fractal detail of the set of singularities is known and the component of the vector field normal to these singularities is sufficiently small. We provide a way of calculating the necessary anisotropic detail for a singular set composed of trajectories. The renormalization theory then gives the existence and uniqueness of solutions to the transport equation (TE), and the corresponding ordinary differential equation (ODE). Further, the trajectories of the flow solution avoid the singular set, which we demonstrated by improving upon the avoidance results of Aizenman and Robinson & Sharples. We retrieve known results in point vortex dynamics in the particular case where the singular set is given by the graphs of a finite number of point vortex Lipschitz trajectories.

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