A Game Theoretical Semantics for a Logic of Formal Inconsistency

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Abstract

This paper introduces a game theoretical semantics for a particular logic of formal inconsistency, called mbC.

Keywords Game theoretical semantics, logic of formal inconsistency, mbC, non-deterministic runs, oracles.

1 Motivation

Paraconsistent logics are the logics of inconsistencies. They are designed to reason *about* and *with* inconsistencies, and generate formal systems which do not get trivialized under the presence of inconsistencies. Formally, a logic is paraconsistent if the explosion principle does *not* hold - in paraconsistent systems, there exist some formulas φ, ψ such that $\varphi, \neg\varphi \nvdash \psi$.

Within the broad class of paraconsistent logics, Logics of Formal Inconsistency (LFIs, for short) present an original take on inconsistencies. Similar to the da Costa systems, LFIs form a large class of paraconsistent logics and make use of a special operator to control inconsistencies (da Costa *et al.*, 2007; Carnielli *et al.*, 2007). Another interesting property of LFIs is that they internalize consistency and inconsistency at the object level. To date, the semantics for LFIs have been suggested using truth tables and algebraic structures. What we aim for in this paper is to present a game theoretical semantics for a particular logic within the class of LFIs. By achieving this, we attempt at both presenting a wider perspective for the semantics of paraconsistency and a broader, more nuanced understanding of semantic games.

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Game theoretical semantics uses game theoretical tools and methods to give meaning to logical formulas. It introduces notions like *strategies* and *wins* to logic. Historically, its roots can be traced back to Lorenz and Lorenzen, yet the modern revival of this approach is mostly due to Hintikka and the Helsinki school (Hintikka & Sandu, 1997; Hodges, 2013).

A *verification game* for a formula φ in a model M checks whether φ is satisfied in M. The verification is achieved by playing an intuitive game. In the classical case, the semantic verification game for classical logic is played by two players, the *verifier* and the *falsifier*. The verifier's goal is to verify the truth of a given formula in a given model. Dually, the falsifier's goal is to falsify it. The rules of the game are specified syntactically based on the current roles of the players. In the game, the given formula is broken into subformulas step by step by the players. A specific instance of the game, a *play*, terminates when it reaches the propositional variables and when there is no move to make. If the play ends with a propositional literal which is true in the model in question, then the current verifier wins the game. Otherwise, the (current) falsifier wins. In classical logic, we associate conjunction with the falsifier, disjunction with the verifier. That is, when the main connective is a conjunction, it is the (current) falsifier's turn to choose and make a move, and similarly, disjunction yields a choice for the verifier. Additionally, in classical logic, negation switches the (current) roles of the players: the player whose role is currently the verifier admits the role of the falsifier and the player whose role is currently the falsifier admits the role of the verifier. The major result of this approach states that the verifier has a winning strategy in the verification game if and only if the given formula is true in the given model. Similarly, the falsifier is said to have a winning strategy if and only if the formula is false in the model. This result is called the *correctness theorem* for game theoretical semantics for classical logic.

Notice that the semantic games in classical logic are very limited game theoretically. They are zero-sum (when one player wins, the other loses), determined (every game has a player admitting a winning strategy), sequential (only one player makes a move at any time), non-cooperative (players compete against each other) and perfect information (the rules and the *game board* are fully known). A recent research program, *paraconsistent game theory*, studies the connection between the aforementioned game theoretical conditions and corresponding logical properties in the context of paraconsistent logics (Başkent, 2015; Başkent, 2016; Başkent, 2017). Which logical rules force determinism in games? Which logics require two-players? Which semantic conditions require non-sequential play? How can we represent the *infectious* truth-values of logics of non-sense in game semantics? So far, this methodology has been applied to a very limited set of non-classical logics. Overall goal of this research program can be viewed in parallel to the program of *logical pluralism* (Beall & Restall, 2006; Priest, 2013). As logical pluralism suggests the ontological possibility of a broader logical toolkit, paraconsistent game theory aims at developing *game theoretical pluralism* where alternative formalisms for games with inconsistencies are suggested. This often requires "stretching and relaxing" certain game theoretical concepts, just as non-classical logics helped us to *redefine* negation both conceptually and formally. As such, the current paper attempts at expanding the domain of semantics games for paraconsistent logics by introducing a new semantics for a particular LFI.

Specifically, in this paper we ask the following questions. How can we give a game theoretical semantics for LFIs? More importantly, following the Hintikkan methodology, what are the game theoretical properties of the verification games for LFIs? By discussing these questions, we will shed light on the connection between LFIs and games. Furthermore, games will provide a somewhat natural explanation for the complicated semantical structure of LFIs, providing an alternative approach. Answering these questions for all LFIs is practically impossible in a single research paper. For that reason, in this paper we discuss a foundational system, called mbC, within the family of LFIs.

The organization of the paper is as follows. First, we introduce mbC - the formal system we focus on, which is followed by a discussion of its semantics. Following, we propose a game theoretical semantics and a set of rules for such a game for mbC. We illustrate some applications of semantic games for mbC, and show the correctness of our game semantics. Before suggesting some possible extensions and alternatives of mbC games, we also present a pragmatic discussion of game semantics with an emphasis on game semantics for mbC. Finally, we conclude with some suggestions for future work. Throughout the text, when we write *game semantics*, we mean *game theoretical semantics.*

2 Logics of Formal Inconsistency and mbC

LFIs extend da Costa systems and generate a broad class of paraconsistent logics (da Costa *et al.*, 2007; Carnielli *et al.*, 2007). In this work, we focus on a particular LFI, called mbC. The system mbC exhibits some of the most important aspects of LFIs. It is "strong enough to contain the germ of classical negation, possessing a kind of hidden classical negation" and contains a consistency operator (Carnielli & Coniglo, 2016).

Let us start with defining the language $\mathcal{L}$ of mbC. Given a set of propositional variables P, we define the syntax of mbC in the Backus-Naur form as follows, where $p \in P$.

$$\varphi ::= p \mid \neg\varphi \mid \circ\varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \to \varphi$$

What distinguishes mbC and other LFIs from other paraconsistent logics is their use of the *consistency* operator $\circ$. The consistency operator simply checks whether a formula *explodes.* This allows us to distinguish and *control* the formulas that can explode the model.

A model M for mbC is a tuple $M = (S, V)$ where S is a non-empty set and $V : \mathcal{L} \mapsto \{T, F\}$ is a valuation function. The function V for mbC assigns a unique truth value to propositional variables, and satisfies the following conditions (Carnielli & Coniglo, 2016):

- $V(\neg\varphi) = F$ then $V(\varphi) = T$,
- $V(\circ\varphi) = T$ then $V(\varphi) = F$ or $V(\neg\varphi) = F$,
- $V(\varphi \to \psi) = T$ if and only if $V(\varphi) = F$ or $V(\psi) = T$,
- $V(\varphi \land \psi) = T$ if and only if $V(\varphi) = T$ and $V(\psi) = T$,
- $V(\varphi \lor \psi) = T$ if and only if $V(\varphi) = T$ or $V(\psi) = T$.

In this semantics, the truth values of $\neg\varphi$ and $\circ\varphi$ are not necessarily determined by the truth value of φ. That is, for instance, if $V(\varphi) = T$, then $V(\neg\varphi)$ is not determined with respect to $V(\varphi)$. It can be either T or F, but not both nor neither. Therefore, the valuation function V is functional, but not truth-functional: mbC valuations are, just as classical valuations, simple functions, but mbC logical operations themselves are *not* functions from tuples of truth values to truth values, but *multifunctions* instead (Carnielli *et al.*, 2007). They assign to the given tuple of truth values a set of possible truth values, from which an mbC valuation is then to pick one for the value of the corresponding complex formula. This valuation is sometimes called *bivaluation*. The semantics for mbC is in sharp contrast to semantics for other paraconsistent logics such as Asenjo's and Priest's LP, which drops the assumption that valuations are functional but keeps classical logical operations essentially intact.

This valuation gives rise to a sound and complete semantics for mbC. We refer the reader to (Carnielli & Coniglo, 2016) for a proof.[1] We give the non-deterministic truth table for some formulas in mbC in Figure 1.

p	$\neg p$	$\circ p$	$p \wedge \neg p$
T	T	F	T
	F	T	F
		F	F
F	T	T	F
		F	F

Figure 1: *Truth table for some formulas in* mbC.

Non-determinacy is one of the complications of mbC and LFIs in general. Next, compared to classical propositional logic, mbC has an extended language with the consistency operator – if a formula and its negation are both true, then the consistency of the formula must be false. Second, perhaps semantically more importantly, mbC assumes that a formula and its negation are *subcontraries*, but not necessarily *contraries*. That is they cannot both be false under the same valuation, but they *can* both be true under the same valuation. It therefore becomes a meaningful question to ask how these complications can be handled using game semantics.

The logic mbC is axiomatized by the following set of axioms.

- $\varphi \rightarrow (\psi \rightarrow \varphi)$
- $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$
- $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$
- $\varphi \wedge \psi \rightarrow \varphi$
- $\varphi \wedge \psi \rightarrow \psi$

[1] As suspected by one of the anonymous referees, the logic mbC does not admit a trivial model — there is no valuation V such that $V(\varphi) = T$ for all formula φ (Carnielli & Coniglo, 2016). If there was one, it would entail that $\circ\varphi$ be false, which can be seen from the truth table of mbC in Figure 1.

- $\varphi \to \varphi \vee \psi$
- $\psi \to \varphi \vee \psi$
- $(\varphi \to \psi) \to ((\psi \to \chi) \to ((\varphi \vee \chi) \to \chi)$
- $(\varphi \to \psi) \vee \varphi$
- $\varphi \vee \neg\varphi$
- $\circ\varphi \to (\varphi \to (\neg\varphi \to \psi))$

The rule of inference is modus ponens: $\varphi, \varphi \to \psi \therefore \psi$.

The last axiom summarizes how mbC *controls* the inconsistencies, whereas all the other axioms are classical. It simply says that only classically-behaved formulas (which satisfy $\circ\varphi$) can explode and trivialize the theory. This axiom is called the *gentle explosion law* in (Carnielli & Coniglo, 2016) as it weakens the principle of explosion.[2]

The logic mbC can be viewed as the starting point for the study of paraconsistent negation, which is also crucial in the game theoretical analysis of non-classical logics. The logic mbC is the *minimal* paraconsistent logic based on classical logic with the *basic* property of *consistency* (hence its name) (Carnielli & Coniglo, 2016). Therefore, the study of mbC and of stronger LFIs based on it are of theoretical interest, as a means toward a deeper understanding of negation and consistency, which has direct relevance not only to paraconsistency but also to intermediate logics and intuitionism. Additionally, as games are used to model interactive and rational situations, game semantics for mbC gives the logic a discussive and argumentative context. This context relates game semantics and paraconsistency to pragmatism. Furthermore, a game theoretical approach to mbC, and in general to non-classical logics, postulates a broader theory of (game theoretical) rationality. How rational agents make moves in semantic games for mbC? Can there exist game theoretical situations similar to mbC semantic games? For a deeper understanding of such issues, a discussion of game semantics for mbC is an essential first step. This is what we achieve in the following.

In this work, we focus on mbC as it is a simple yet powerful logic within the family of LFIs. For our purposes, mbC is an ideal candidate. Its semantics is rather different from the various other non-classical logics as it resorts to bivaluations. It has a special operator to express the meta-logical concept of consistency at the object level. From a practical perspective, mbC is also a well-studied example of LFIs, and relates directly to many other non-classical logics – including fuzzy, intuitionistic and minimal logics. Furthermore, mbC, along side with many other LFIs, interacts with the classical propositional logic in an interesting way. As underlined by Carnielli and Coniglio, mbC is both a subsystem and an extension of classical logic (Carnielli & Coniglo, 2016). This allows us to discuss the nuances of game semantics, which was initially suggested for classical logic, in a broader formal framework. It is also important to note that mbC is a propositional system. Focusing on the game semantics of mbC, therefore, generates a fruitful project with the potential of expanding it to

[2]As we mentioned earlier, the lack of the principle of explosion is a defining property for paraconsistent logics. In LFIs, on the other hand, the explosion is "gentle" and fails *only* for certain formulas.

the modal and first-order extensions of mbC. This is how we justify our choice for mbC.

So far is the basics of mbC. In what follows, we present a game semantics for mbC.

3 Game Theoretical Semantics for mbC

The game semantics for mbC faces two immediate challenges. The first challenge is to give a game theoretical interpretation for the consistency operator $\circ$. The second one is to design a semantic game with *semantic* non-determinism in the sense that the negation of a true formula can be either true or false non-deterministically.

An interesting aspect of game semantics is that it is not compositional. A compositional semantics is a system where the truth value of complex formulas is determined based on the truth value of their subformulas. Algebraic semantics and truth-tables offer compositional semantics. In semantic games, however, players make choices without *testing* the alternatives. The truth value of the formula, thus, depends only on the choices made, not in addition to the choices *not* made.

Game semantics for mbC requires a relaxation of the non-compositionality principle. As we mentioned earlier, the $\circ$ operator internalizes the meta logical concept of consistency at the object level in the logic, and it is effectively a consistency-checker. In order to give a game theoretical semantics for the behavior of the $\circ$ operator, the truth values of both a formula and its negation need to be considered at the same time (see Table 1). This will generate two subgames for both possible choices. We will use this observation when we discuss the game rules for the $\circ$ operator.

On the other hand, our approach to non-determinism is similar to the non-deterministic choices in the dynamic logic of programing (Harel *et al.*, 2000). In the propositional dynamic logical syntax, the formula $[a \cup b]p$ means that after a non-deterministic run of programs a or b, the proposition p holds. In this case, it remains as a non-deterministic choice which program is run. In propositional dynamic logic, for the expression $[a \cup b]p$, where a, b are programs and p is a proposition, we have the following equivalence:

$$[a \cup b]p \equiv [a]p \wedge [b]p$$

That is if p is the output of a non-deterministic run of programs a and b, this means that both programs, independent from each other, should produce the output p. Therefore, from a computational point of view, a non-deterministic run reduces to two different programs, a and b in this case, running with the same input and producing the same output. In a non-deterministic choice, both choices should be ensured to produce the same output - simply because it is not determined which program will be executed. Notice that this is not just an epistemic ambiguity, it is non-determinism.

We adopt a similar approach in this work. In a game, using the logical vocabulary, if a player can non-deterministically win a game, this amounts to the fact that he has individual wins in each non-deterministic choice.[3] His strategy must work in both cases.

[3] As one of the anonymous referees observed, our computational reading of non-determinacy is rather different than the standard notion of non-deterministic wins in game theory. Our semantic games do not

This point underlines our philosophical motivation. We classify our philosophical motivation into two main categories: (semantic) non-determinism of truth and constant-sum semantic games. Non-determinism of truth is formally complex and it gets more complicated within the context of paraconsistency. More importantly, use of game semantics in understanding non-determinism in paraconsistency directly relates to the historical roots of paraconsistency. Jaśkovski's discursive logic, suggested in 1948, postulates and motivates paraconsistency within an interactive discussive context (Jaśkowski, 1999). In his suggestion, paraconsistency is viewed as an inconsistency within the beliefs and opinions of the discussants. The key point of Jaśkovski's approach is to motivate it in an implicit multi-agent context. In such context, non-determinism is a natural paradigm: participants (discussants, agents, players) may assert p without having any opinion about $\neg p$. What is the logic of such situations? Particularly, what is the *paraconsistent* logic of such *interactive non-determinism*? Game semantics for LFIs suggests an answer to this question by remaining loyal to the Jaśkovskian roots of paraconsistency. Such a discussion on the discussive roots of paraconsistency relates the subject to pragmatism as well, which we discuss later on in detail.

Second, semantic games often enjoy simple game theoretical properties, as we underlined earlier. A philosophical contribution to this debate should start with classifying the semantic games for different logics: which semantics games for which logics are constant sum? Conversely, semantic games suggest new tools to "develop" logical systems with the desired properties for constant-sum games. Our work presents a first step towards this direction.

Now, we start by describing the semantic games for mbC. First of all, the game G is played by two players: the verifier (**V**) and the falsifier (**F**). In the literature, sometimes these players are called *Heloise* and *Abelard* or *me* and *Nature*, respectively. We denote the set of players by Π.

The game G is played on a model $M = (S, V)$ where S is a non-empty set and $V : P \mapsto \wp(S)$ is a valuation function defined on the set of propositional variables P and extended to all the formulas in $\mathcal{L}$ in the standard way. The model can be viewed as the *game-board* or the *arena* of the verification game. A given formula φ is evaluated in the model M by playing a game to see whether $M \models \varphi$ or not.

We allow the set of positions Σ vary over the language: $\Sigma \subseteq \{(\pi, \varphi) : \pi \in \Pi, \varphi \in \mathcal{L}\}$. The set Σ is precisely determined based on the given formula, players' roles, underlying logic and the game rules. The algorithm which is used to determine Σ in classical game semantics works in our case as well. Hence, we will not worry about the details of this procedure here.

A (pure) *strategy* for player i is a set of rules that tells him which move to make at each position where it is his turn. A *winning strategy* for i is the one that guarantees a win for i regardless of the moves of the opponent(s).[4] The correctness theorem of

contain any probabilistic choice at all. Our reading of non-determinism is rather similar to that of non-determinism in automata theory.

[4]As one of the anonymous referees underlined, this is a critical definition. Winning strategies are (traditionally) defined for win/lose games. Otherwise, the standard terminology is "strategy guaranteeing maximal payoff". But, this is problematic as well especially within the context of (classical) game semantics, where there is no (ordinal or otherwise) payoff. Keeping these points in mind, we choose to use the tradi-

game semantics postulates how the existence of winning strategies relates to the truth of the formula in question in semantic games. However, in general, it is very much possible to have semantic games that are not win/lose. In that case, we may have games where one player's win may not amount to the opponent's loss.[5]

The winning conditions for mbC games expand those for the semantic games for classical logic. The current verifier wins G if the game ends with a propositional variable which is true in the model and there is no more move to make. Dually, the current falsifier wins if the game ends with a propositional variable which is false in the model and there is no more move to make.[6]

In game semantics for mbC, we allow players to have multiple roles in terms of being verifiers or falsifiers. Let us then introduce some notation for clarity. By $\mathbf{X}_{\mathrm{x}}$, we denote the situation where player $\mathbf{X}$ has the role x. The player $\mathbf{X}$ varies over Π by definition, and similarly the role x can be either v or f which stand for the verifier and falsifier, respectively. In particular, $\mathbf{V}_{\mathrm{v,f}}$ denotes the cases where player $\mathbf{V}$'s role is *undetermined*. In this case, due to our reading of non-determinacy which we explained earlier, the player $\mathbf{V}$ assumes *both* roles - his strategy must work in either case. For simplicity of notation, when players assume their default roles, we will not specify it with a subscript. We will refer to this notation especially when we discuss negation and the associated game rule, where players switch roles.

Similar to game semantics for some other non-classical logics, the game semantics for mbC requires some changes in the standard verification game (Başkent, 2015). The change we propose is to have a constant sum semantic verification game, reflecting the non-deterministic and paraconsistent character of mbC. In our games, we allow more than one player to have the same role.

We can now formally define the verification game G for mbC.

Definition 3.1. A mbC verification game is a tuple $G = (\Pi, \Sigma, \rho)$ where Π is the set of players *verifier* $\mathbf{V}$ and *falsifier* $\mathbf{F}$, Σ is the set of positions and ρ is the set of well-defined game rules.

A specific play of G on M for a formula $\varphi \in \mathcal{L}$ is denoted by $G(M, \varphi)$. Similarly, the set of positions Σ_φ in $G(M, \varphi)$ is restricted to the subformulas $\mathsf{Sub}(\varphi)$ of φ. Precisely, $\Sigma_\varphi = \{(\pi, \psi) : \pi \in \Pi, \psi \in \mathsf{Sub}(\varphi)\}$. Which player π with what role is associated to which subformula ψ is determined by the game rules following the standard methodology of classical game semantics.

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tional terminology with a twist towards our goal of establishing paraconsistent game theory. The winning strategies in semantics games for non-classical logics are constructed based on the winning conditions in such games, which can be non-classical logically.

[5] This certainly requires a different, perhaps non-biconditional, correctness theorem for the game semantics of such systems. For certain examples of such games and logics, we refer the reader to (Başkent, 2016).

[6] This is the point where wins are defined reflecting the behavior of negation. The winning conditions in classical logics necessarily require win/lose games. However, if "truth in a model" is defined paraconsistently, then more than one player may have wins, depending on the model theory of the paraconsistent logic in question. Consequently, winning strategies are defined based on such a definition of wins and constructed for games that are not necessarily win/lose games — as we have done. Nevertheless, as we mentioned earlier, we stick to the traditional terminology in this work in order to prevent an inflation in terminology and also to underline the fact that the concepts in semantic games for classical logics are degenerate cases of those in semantic games for paraconsistent logics.

Now, our main task is to give the game rules ρ for G. These rules will be defined as transformations from a game position (π, φ) to a set of positions $\{(\pi', \psi)\}$ where $\pi' \in \Pi$ and ψ is a subformula of φ defined in the standard way.

The game rules ρ for semantic games for mbC are given as follows based on the players' *current* roles. Similar to many other semantic games, distribution of roles (verifier, falsifier etc.) may change throughout the game in mbC verification games.

(ρ^G_p) If the current formula is p for $p \in P$, the current verifier wins if p is true and the current falsifier wins if p is false,

($\rho^G_\neg$) If the current formula is $\neg\varphi$, then the current falsifier becomes the verifier, and the current verifier becomes both the verifier and the falsifier, and the game continues with φ with roles switched,

($\rho^G_\circ$) If the current formula is $\circ\varphi$, then the game continues with φ and $\neg\varphi$ with players' roles switched,

($\rho^G_\to$) If the current formula is $\varphi \to \psi$, then the current verifier makes a choice between $\neg\varphi$ and ψ,

($\rho^G_\wedge$) If the current formula is $\varphi \wedge \psi$, then the current falsifier makes a choice between φ and ψ,

($\rho^G_\vee$) If the current formula is $\varphi \vee \psi$, then the current verifier makes a choice between φ and ψ.

Game rules specify which moves are allowed when a game is in a certain position and which players make moves. It is important to notice that the rules are given for the *current roles* of the players. During the game players may switch roles back and forth multiple times. For example, the falsifier can start the game as $\mathbf{F}$, can continue as $\mathbf{F}_{\mathrm{v}}$ as a verifier, and terminate the game as $\mathbf{F}_{\mathrm{v,f}}$ where he can end up admitting both roles. In that respect, simply put, the game rules are given for the "subscripts" of the players, which indicate the current role of the player.[7]

The set of game rules ρ follows closely the tableaux rules for mbC (Carnielli *et al.*, 2007). It also reflects our reading of non-determinism.[8]

Let us now see some examples. We start with negation.

[7] It is often found counter-intuitive that the players and roles are switched throughout semantic games. Even if it is an entirely different discussion which falls outside the scope of this paper, there is a simple solution. For any given formula φ, it is possible to play semantic games for the negation normal form of φ. In this way, the role distribution for the game takes place at the beginning and does not change during the game. The downside of this methodology is that, syntactically, the play for φ and its (logically equivalent) negation normal form are not necessarily identical. Moreover, not all non-classical logics satisfy the De Morgan principles, thus may not admit a negation normal form (Ferguson, 2012).

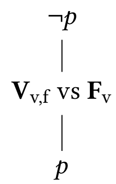
[8] This is indeed one of the tableaux rules for mbC where we have the following.

$$\frac{}{\mathrm{T}(\varphi) \mid \mathrm{F}(\varphi)}$$

This rule is necessary for the tableau to close. Our game rules admit no corresponding rule, and this is what leads to both players having winning strategies for certain formulas.

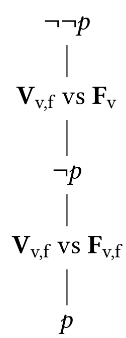
Example 3.2. Consider $\neg p$ for propositional variable p. Let us see how game theoretical semantics works in this case.

If p is true, then the truth value of $\neg p$ is undetermined and can be either. Thus, we expect both players to have winning strategies. This is indeed the case as both $\mathbf{V}_{\mathrm{v,f}}$ and $\mathbf{F}_{\mathrm{v}}$ wins when p is true.



On the other hand, if p is false only the falsifier, that is $\mathbf{V}_{\mathrm{v,f}}$, will win the game. Thus, she will win the game for $\neg p$ as the negation of a true formula is deterministically false in mbC.

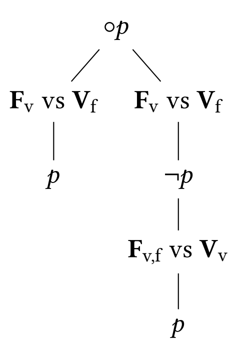
Example 3.3. Let us consider the game for $\neg\neg p$ and see how game theoretical semantics works in this game.



In this case, various unusual valuation may appear. For example, if all p, $\neg p$ and $\neg\neg p$ are true, then the verifier still wins the game for $\neg\neg p$. If $\neg\neg p$ is false, but $\neg p$ and p are true, the falsifier still has a winning strategy for the game for $\neg\neg p$ as the game tree demonstrates.

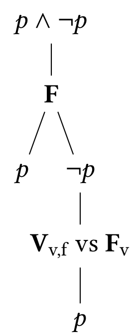
The analysis of other cases easily follows from the game tree.

Example 3.4. Let us consider the case $\circ p$.



An interesting case is where both p and $\neg p$ are true, but $\circ p$ is false. In this case, as expected, on the left-hand side branch $\mathbf{F}_{\mathrm{v}}$ has a winning strategy, yielding a winning strategy for $\mathbf{F}$ for the formula $\circ p$. On the right-hand side branch, $\mathbf{F}$ has a winning strategy, too - first making a move as $\mathbf{F}_{\mathrm{v}}$ and as $\mathbf{F}_{\mathrm{v,f}}$.

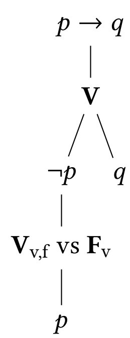
Example 3.5. Let us consider the case $p \wedge \neg p$.



The only interesting case is where both p and $\neg p$ are true, yielding $p \wedge \neg p$ true.

In this game, **F** loses immediately if he chooses p. If he chooses $\neg p$, then $\mathbf{V}_{\mathrm{v,f}}$ makes a choice and wins the game for $p \wedge \neg p$. Additionally, the falsifier makes a move as $\mathbf{F}_{\mathrm{v}}$ and wins. In this game, both players have winning strategies.

Example 3.6. Let us consider the case $p \rightarrow q$. The problematic case in mbC is when both p and $\neg p$ are true, and q is false. In this case, the conditional $p \rightarrow q$ is false.



In this case, we will show that **F** has a winning strategy.

If **V** chooses q, which is false, then **F** wins.

If **V** chooses $\neg p$, then the game continues with p with the players $\mathbf{V}_{\mathrm{v,f}}$ and $\mathbf{F}_{\mathrm{v}}$, where **F** wins again.

Since the game is not assumed to be zero-sum, **V** can win as well.

As the examples indicate, the existence of multiple winning strategies does not conclusively determine the truth value of the formula in question. This is simply a reflection of the truth conditions of mbC, and separates mbC games from many other non-classical logical semantics games studied in (Başkent, 2016).

We can now give the correctness theorem for the game semantics for mbC.

Theorem 3.7. *In an mbC game $G(M, \varphi)$ for a model M and a well-defined formula φ,*

- *The verifier has a winning strategy if φ is true,*
- *The falsifier has a winning strategy if φ is false.*

Proof. The proof is by induction on the complexity of φ.

Case 1: Propositional Atoms

The case for the propositional atoms is identical to the classical case and follows from the game rules.

Case 2: Negation

Let φ be $\neg\psi$ for some formula ψ, and assume that $\neg\psi$ is true. We will show that the verifier **V** has a winning strategy in $G(M, \neg\psi)$. By the game rule $(\rho_{\neg}^{G})$, we have $\mathbf{V}_{\mathrm{v,f}}$ and $\mathbf{F}_{\mathrm{v}}$ in $G(M, \psi)$. According to the truth table for mbC, if $\neg\psi$ is true, then ψ can be either true or false. Thus, in each case, $\mathbf{V}_{\mathrm{v,f}}$ can win the game $G(M, \psi)$, and he is the only player who can do that according to the game rules and win the game in each case. Then, in $G(M, \varphi)$, $\mathbf{V}$ simply follows the rule and plays his strategy in $G(M, \psi)$. This constitutes his winning strategy in $G(M, \varphi)$.

This was the proof for the verifier, let us now see how the proof works for the falsifier $\mathbf{F}$. The key observation here is the fact that in mbC, if $\neg\psi$ is false, then ψ must be true (as opposed to undetermined).

Let $\neg\psi$ be false. We will try to show that the falsifier $\mathbf{F}$ has a winning strategy in $G(M, \neg\psi)$. Now, by the truth table, ψ must be true. In $G(M, \psi)$, we have $\mathbf{F}_{\text{v}}$ and $\mathbf{V}_{\text{v,f}}$ as players. By the induction hypothesis, verifiers have winning strategies. The player $\mathbf{F}_{\text{v}}$ is one of the verifiers. Since $\mathbf{F}_{\text{v}}$ has a winning strategy in $G(M, \psi)$, she has a winning strategy in $G(M, \neg\psi)$ - she simply follows the game rule ($\rho^G_\neg$) and makes a move as a verifier in $G(M, \psi)$. Notice that the player $\mathbf{V}_{\text{v,f}}$ also admits a verifier role in $G(M, \psi)$, but this does not exclude the player $\mathbf{F}_{\text{v}}$ admitting a winning strategy as we did *not* assume that the verification game is zero-sum.

This concludes the proof for negation.

Case 3: Consistency Operator

Let φ be $\circ\psi$ for some ψ. We start with the case for the verifier.

Let us assume that $\circ\psi$ is true to show that the verifier $\mathbf{V}$ has a winning strategy in the game $G(M, \circ\psi)$. Now, by the truth table, either ψ or $\neg\psi$ is false - not both, not neither. Without loss of generality, assume it is ψ. Thus, $\mathbf{F}$ has a winning strategy in the game $G(M, \psi)$. Then, by rule ($\rho^G_\circ$), $\mathbf{V}$ has a winning strategy in the game $G(M, \circ\psi)$, hence in $G(M, \varphi)$.

Let us now see the case for the falsifier $\mathbf{F}$. Let us assume that $\circ\psi$ is false. Then at least one of ψ and $\neg\psi$ is true. By the induction hypothesis, the verifier in $G(M, \psi)$ or $G(M, \neg\psi)$ has a winning strategy. Thus, by rule ($\rho^G_\circ$), the falsifier $\mathbf{F}$ has a winning strategy in $G(M, \circ\psi)$ - hence in $G(M, \varphi)$.

Cases 4 & 5 & 6: Conditional, Conjunction and Disjunction

The cases for conditional, conjunction and disjunction are skipped as they are identical to the classical cases.

This concludes the proof. ∎

The proof shows that in the semantic games for mbC, one player's loss or win does not guarantee the opponent's win or loss. It further clarifies the game theoretical reading of non-determinism. In this approach, non-determinism can be viewed as super-strategizing where strategies include all choices.

The converse of the Theorem 3.7 is not true. Because in some games players may admit multiple roles. As the rule ($\rho^G_\neg$) shows, the verifier can admit both roles in a play, reflecting the non-deterministic character of the game. Therefore, the existence of the winning strategies does not determine the truth value as a player may admit winning strategies for both roles.

Moreover, the converse of Theorem 3.7 cannot be established by imposing various further restrictions on the existence of winning strategies, including those mentioned in an earlier paper (Başkent, 2016). In (Başkent, 2016), for example, the converse statements of the correctness theorems for some logics were given by imposing "uniqueness" conditions on the existence of winning strategies for certain players. For instance, it was stated that if *only* the player, whose current role is to force the game to an end with the paraconsistent truth value P in Priest/Asenjo's Logic of Paradox, has a winning strategy in the verification game, then the truth value of the formula in question has to be P. This methodology, however, does not work in mbC. In order to see this, assume that in a game $G(M, \neg\psi)$ only the falsifier has a winning strategy. Then, the game continues with $G(M, \psi)$ where we have the players as $\mathbf{F}_{\mathrm{v}}$ and $\mathbf{V}_{\mathrm{v,f}}$. That is, the falsifier admits the role of a verifier. Yet, the verifier also admits the role of a verifier (in addition to its falsifier role) in $G(M, \psi)$. Therefore, the idea of "imposing the uniqueness of the falsifier's winning strategy to obtain a unique truth value for the formula" collapses. If both players have the same role, whose role will determine the actual truth value of the formula?

We proceed with the following results.

Corollary 3.8. *A mbC game $G(M, \neg\varphi)$ for a model M and a well-defined negation-free formula φ may not be won by only one player.*

Proof. Consider the situation where φ is true, rendering its negation undetermined. In this case, both players have winning strategies. ∎

Notice that the existence of winning strategies for both players does not refute Theorem 3.7.

Corollary 3.9. *In a mbC game $G(M, \varphi)$ for a model M and a well-defined formula φ, both players can win the game. But, it is not possible that both can lose the game.*

Proof. The result directly follows from the game rules. ∎

The above result establishes that the semantic games for mbC are not zero-sum. From a game theoretical perspective, it suggests a direct connection between certain bivaluations and the "number" of winners in certain games.

4 Discussion: Pragmatics of Game Semantics for mbC

The most obviously distinctive feature of game-theoretical semantics is that it postulates an interactive and dialogical argumentative situation, as opposed to the essentially monological form of discourse of mathematical proofs (Rahman & Carnielli, 2000). Game-theoretical semantics, therefore, allows for actual conceptualization of truth and falsity as features of certain well-defined interactive situations - namely, the existence of winning strategies. Then, pragmatically, for which discursive situations, if any, do semantic games stand?

This dynamics clearly resembles a debate of sorts. In a debate, or at least in one in which *truthful* discourse is to be expected, the person who makes a claim is bound to stand for it, effectively committing to its truth, and whoever objects to the claim commits to its falsity; and, of course, usually both parties also want to be *right* and, thus, win the debate.

Similarly, in semantics games, by committing to the truth (or falsity) of φ, a player, if *coherent*, implicitly assumes certain other commitments which are determined by the subformulas of φ. Classically, for example, if φ is a conjunctive formula $\alpha \wedge \beta$, then, since it is necessary for the truth of $\alpha \wedge \beta$ that both α and β be true, the one who commits to the truth of φ -the verifier- implicitly commits to the truth of both α and β, while the one who commits to the falsity of φ -the falsifier-, needs to commit to the falsity of only one of them. Since, however, her pragmatic commitments do not tell him to *which* of α and β to commit, the falsifier may do so as he prefers.

Notice, however, that the relevant form of commitment here is *pragmatic*, and not *epistemic*. In a debate one may just as well stand for something they don't believe - indeed, lawyers do this sort of thing all the time. We may say, with Brandom (Brandom, 1994), that rational debate is a matter of what claims one is *entitled* to - as opposed to the beliefs one actually holds in private. Classically, if one wants to claim that φ, and even more so if one does not share any assumptions with the interlocutor from which the formula could be justified inferentially, then there is no way around it but to dispute that entitlement (and win). Ultimately, this is to say that one must have a winning strategy as the verifier in the semantic game for a formula in order to be sure to be entitled to claim it. In a sense, thus, semantic games simply spell out the rules of propositional entitlement.

This being so, we may ask: how does semantics games for mbC contribute to the discussions about the pragmatics of LFIs, especially considering its non-deterministic aspect?

What is interesting about mbC (and LFIs in general) is that, unlike Asenjo's and Priest's LP, for example, they are paraconsistent logics which *do* have the means to force contrariety of a formula φ and its negation, thus recovering classicality. Indeed, this is the very upshot of adding a consistency operator to the object language. When one is not sure of the consistency of φ, however, $\neg\varphi$ may be compatible with both φ being true and it being false. It thus follows, in particular, that there are some cases in which one needs not (as would be the case in classical logic) be entitled to object to φ in order to be entitled to claim that $\neg\varphi$.

As expected, the proposed game semantics for mbC mimics this behaviour by making it at least possible for the verifier of $\neg\varphi$ to win (and, indeed, also have a winning strategy), whether or not the falsifier has a winning strategy. Now by going back to pragmatics, we see that mbC and other LFIs would be most properly conceived as modelling standards of inference of situations in which opposing parties may be equally entitled to *prima facie* incompatible claims. This is precisely what is to be expected of situations in which the agent's assessment of the truth of falsity of the claims is to be bounded by currently available evidential support. In such a situation, the one who is committed to the truth of an actually true proposition, for example, may nevertheless be no more (nor less) entitled to claim it than her opponent in the dispute, if the available evidential support is not decisive.

In general, this seems to imply that non-determinism in the semantics of a paraconsistent logic is better understood in its potential relation to epistemic game theory, particularly to *informationally bounded rationality*. As the term itself suggests, non-determinism in games relates directly to how game theoretical rationality is conceived with its relation to information. In long term, this suggests that the connections between LFIs and game theory are far more reaching than previously envisioned. On the one hand, one of the most central challenges presented to game theory today is imperfect information games and how they related to rationality and game theoretical equilibria (Osborne & Rubinstein, 1994); and LFIs may be just the logical tools needed to aid game theory in such a task, by providing it with a systematic way of approaching informational constraints. On the other hand, different informational constraints, motivated by various game theoretical situations, would provide philosophical interpretations (and, more importantly, applications) for a variety of LFIs. This would ground LFIs and at the same time present a non-classical logical connection to game theory.

We started our discussion of game semantics for mbC with an analogy between non-determinism in program runs and non-determinism in truth values. This, however, turns out to be more than an analogy.

From a computational perspective, the game semantics for mbC is significant and contributes to the discussion regarding the constructive nature of proofs. If proofs are considered as realization of winning strategies, and if the truth is identified by (the existence) proofs, non-determinism simply builds upon this connection. In mbC-games, it is then possible to view the existence of winning strategies as proofs for non-deterministic truth. Therefore, it remains as a task for future work whether this connection can be extended to various major discussions in computational logic, including non-deterministic automata and P vs NP problem. By introducing non-determinism to semantic games, thus to proofs, we take a small steps towards understanding the logical foundations of non-determinism in computation in general. From a pragmatics point of view, this simply shows us whether computational choices can be seen as pragmatic choices and *vice versa*. The pragmatic connection between computation, games and non-determinism falls beyond the scope of this paper, and we leave this discussion to future work.

5 Further Observations

The computational approach to mbC is more than an analogy. It suggest further connections between pragmatically motivated philosophical approaches and game theoretical analysis of mbC. Following the computational path, we suggest handling non-determinism using a well-known concept in computability theory. Next, following the game theoretical path, we relate mbC games to logical dependancy. Logical dependency is a central concept in logic of games and it can be used to formalize imperfect information games.

5.1 Oracle

An oracle is a know-it-all meta-player that can be asked any question and answers truthfully. For that reason, he can be viewed as a *consistency-checker*, where he simply answers for which formula φ, $\circ\varphi$ is true. With an oracle, it is then possible to reduce mbC games $G(M, \varphi)$ to classical games where $\circ\varphi$ holds. However, if the oracle answers negatively about the truth of $\circ\varphi$, then the game continues as before. Our aim is to suggest a computational tool which can express the gentle explosion in mbC using game semantics.

The new game rules are given as follows for semantic games with oracles.

($\bar{\rho}_p^G$) If the current formula is p for $p \in P$ and $\circ p$ is true, then the verifier wins if p is true and the falsifier wins if p is false,

($\bar{\rho}_\neg^G$) If the current formula is $\neg\varphi$ and $\circ\varphi$ is true, then the game continues with φ with players switching roles,

($\bar{\rho}_\circ^G$) If the current formula is $\circ\varphi$ and if $\circ\varphi$ is true, the game continues with φ.

The rules for conjunction, disjunction and conditional remain the same.

The semantic games where the oracle answers each question affirmatively is called a *verification game with an oracle*, and denoted by $\overline{G}(M, \varphi)$ for a given formula $\varphi \in \mathcal{L}$ and a model M. Those are the games that benefit fully from an oracle. In such games, we have a stronger correctness theorem.

Theorem 5.1. *In a mbC verification game $\overline{G}(M, \varphi)$ with an oracle in a model M for a given formula $\varphi \in \mathcal{L}$,*

- *The verifier has a winning strategy if and only if φ is true,*
- *The falsifier has a winning strategy if and only if φ is false.*

Proof. The proof is by induction on the complexity of φ.

Case 1: Propositional Atoms

If for a $p \in P$, we have $\circ p$ true, then by the truth table p and $\neg p$ cannot both be true. Then, the game is classical and the classical winning conditions apply, hence the result follows.

Case 2: Negation

Similarly, in this case, according to the truth table in Table 1, the game reduces to the classical case and the classical game rules apply, and the result follows.

Case 3: Consistency Operator

This case shows the over-reaching strength of the oracle.

If for $\varphi = \circ\psi$ for some ψ and $\circ\psi$ is true, then the game continues with ψ classically. What we learn from the oracle is that ψ behaves classically, so is φ. As such, the operator $\circ$ becomes redundant. The game continues with ψ classically.

Cases 4 & 5 & 6: Conditional, Conjunction and Disjunction

Follows immediately using the classical truth conditions.

This concludes the proof. ■

The immediate gain for introducing an oracle is regaining the zero-sumness of the verification game.

Corollary 5.2. *mbC verification games with oracles are zero-sum.*

It is important to notice that the oracle is a meta-player, it does not admit winning conditions and cannot win the game, but it partially resolves the non-determinacy of the truth values in some cases.

As we mentioned, the new game rules $\bar{\rho}$ can be seen as too strong. There are cases where a formula φ behaves classically but $\circ\varphi$ is not true. This is indeed a justified criticism. The new game rules allow the oracle to overplay his hand and reduce the game to a simple sub-game where the biconditional correctness theorem can easily be recovered. This is a reflection of the truth table for mbC.

In LFIs, the meta-logical consistency property is brought down to the object level by the $\circ$ operator. The oracle is defined on this observation: it answers meta-logical questions and communicates the answer in the object level. It is easy to see that different oracles could be implemented for the non-classical connectives in mbC. For example, it is possible to define an oracle that can answer questions about the consistency of $\{\varphi, \neg\varphi\}$ with the aim of resolving the truth value of the negation. Such extensions are important as they beg further computational questions: What is the minimum number of oracles necessary in order to obtain a biconditional correctness theorem for a given LFI? How can we reduce different LFIs to each other using oracles? Such questions are interesting and point out various further research opportunities, motivated by the game semantics.

Our oracle is omniscient. It is possible to consider another oracle who is *omnipotent*. The omnipotent oracle can have the power of turning the mbC formulas φ into classical ones by ensuring that $\circ\varphi$ is satisfied for such φ.

Furthermore, the oracle is a know-it-all player. Then, the immediate question is how the oracle plays in imperfect information games, which can possibly be formalized by semantic games for mbC. In such a case, the role of the oracle crystallizes the epistemic aspects of the game.

In conclusion, the introduction of oracles to semantic games suggests an alternative approach to understand the relation between LFIs and classical logics. In addition to semantical, proof-theoretical and algebraic methods which are suggested to analyze this relation, our contribution broadens the discussion to a computational and epistemic framework. This should be seen as another step towards understanding the connection between proofs, game semantics, non-classical logics and strategies.

5.2 Dependence

Another direction to extend the framework we have introduced is to consider a relatively recent body of work which focuses on dependence and independence of quantifiers (Mann *et al.*, 2011; Väänänen, 2007). These logics introduce the idea that in a game, sometimes a player's move may or may not depend on the opponent's prior moves. The quantifier dependence henceforth can be viewed as a game from this perspective.

It is possible to take another step towards this conceptual direction and discuss dependency of truth-values within the context of LFIs, different from how dependency is discussed in IF logic. The truth value of a composite formula in some cases does and in some cases does not, depend on the truth value of its consistency. Such a dependence requires considering the negation of the formula in question. Simply put, the truth value of $\neg p$ does *not* depend on the truth value of $\circ p$ if p is false. However, it *does* depend on the truth value of $\circ p$ when $\circ p$ is false. Moreover, when the truth value of $\neg p$ does *not* depend on the truth value of $\circ p$ when p is true, then it can be deduced that $\neg p$ is false.

As we have observed in the proof of the Correctness Theorem (Theorem 3.7), some truth values non-deterministically create dependency. For example, if a negation $\neg p$

is false, then due to the semantics of mbC, $\neg p$ cannot be *only* false, it must be non-deterministically false. In this case, we observe that the existence of a winning strategy for the verifier depends on the existence of a winning strategy for the falsifier in $G(M, \neg p)$. This is an original approach to the dependency of winning strategies in verification games and suggests a complex way to strategize for the players. We summarize these observations briefly as follows.

Proposition 5.3. *In a mbC verification game, the existence of a winning strategy for one player may depend on the existence of a winning strategy for the opponent in some subgames.*

Notice that the above proposition is another way to conclude that the semantic games for mbC are not zero-sum but positive-sum.

Proposition 5.4. *The verification game for mbC is a non-negative sum game.*

Therefore, the game rules and the dependence of strategies both suggest that the semantic games for mbC are constant-sum. What is left is to extend this approach to a broader context where strategy dependence and its relation to underlying logic (whether it is mbC or some other LFI) is studied. That is for future work.

6 Conclusion

Non-determinism is not a very intuitive concept neither for theories of truth nor for game theory. Coupled with paraconsistency, it becomes even more complex. Game semantics aims at clarifying this issue. It explains how non-determinism works, how it should be understood and how it relates to other computational concepts and games, such as oracles and imperfect games. This provides further evidence to the connection between proofs, winning strategies and truth.

What we achieved in this work, motivated by various computational notions and logics, is to introduce non-determinism to Hintikkan semantic games and present a new semantics for LFIs. Consequently, we established a grounded application of non-determinism in propositional dynamic logic in semantic games, and used a know-it-all meta-player, the oracle, to normalize the semantic games. Both of these directions are promising, relating a broader class of games and logics.

LFIs form a broad category of logics. In this work, we focused on one of the minimal LFIs. What remains to be done is to extend the current work to other LFIs in a systematic way. One way is to consider the distribution of roles as suggested by the negation operator. In our system, negation introduces a second role for the verifier. Considering various alternatives for the distribution of roles over players under negation suggests a procedural way to extend the game semantics we have developed to some other logics within the LFI system. Furthermore, it also offers a way to "develop" logics based on game rules without focusing on the truth table first. Another line for future work is to extend the game semantics to other non-deterministic logics via quasi-matricies and swap structures (Avron & Zamansky, 2011). We leave such generalization for future work.

The remarks we presented on the pragmatics of game semantics easily carry over to various neighboring fields of game theory, including social choice theory and welfare theory. Social choice theory, in particular, has been shaped by impossibility results and paradoxes. This presents immense opportunities to apply paraconsistent game theory to the impossibility results in social choice theory. An inquiry regarding the possible applications of LFIs in social choice theory as well as seeking solutions from LFIs for the classical problems of the theory remains as valid direction for further study.

Game semantics for logics are not necessarily unique. Seen as mathematical objects, game equivalence is not an easy concept to define. What follows from this observation is that there can exist various other semantic games for mbC which potentially treat the consistency operator and negation differently using alternative game theoretical notions. It is our aim that the current work stimulates such developments in the areas of semantic games and non-classical logics.

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