COMPACT COMPOSITION OPERATORS WITH SYMBOL A UNIVERSAL COVERING MAP ONTO A MULTIPLY CONNECTED DOMAIN

MATTHEW M. JONES

Abstract. We generalise previous results of the author concerning the compactness of composition operators on the Hardy spaces H^p, $1 \leq p < \infty$, whose symbol is a universal covering map from the unit disk in the complex plane to general finitely connected domains. We demonstrate that the angular derivative criterion for univalent symbols extends to this more general case. We further show that compactness in this setting is equivalent to compactness of the composition operator induced by a univalent mapping onto the interior of the outer boundary component of the multiply connected domain.

1. Introduction

Let $\mathbb{D} = \{z \colon |z| < 1\}$ be the unit disk in the complex plane and H^p the classic Hardy space of holomorphic functions f on $\mathbb{D}$ satisfying

$$\|f\|_p^p = \lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

If $\phi \colon \mathbb{D} \to \mathbb{D}$ is a holomorphic mapping then the composition operator

$$C_\phi \colon f \mapsto f \circ \phi$$

is well defined and maps H^p boundedly into itself for any $0 < p < \infty$.

Compactness of C_ϕ, in contrast, depends on ϕ in a more subtle and interesting way. It was shown in [6] that C_ϕ is compact on H^p, $1 \leq p < \infty$, if and only if

$$\lim_{|w| \to 1} \frac{\mathcal{N}_\phi(w)}{\log 1/|w|} = 0$$

Date: April 6, 2016.

1991 *Mathematics Subject Classification.* Primary 47B33; Secondary 30F35.

Key words and phrases. Composition operator, Hardy space, Fuchsian group, Poincare series.

where $\mathcal{N}_\phi$ is the Nevanlinna counting function

$$\mathcal{N}_\phi(w) = \begin{cases} \displaystyle\sum_{\phi(z)=w} \log \frac{1}{|z|}, & w \in \phi(\mathbb{D}) \\ 0, & \text{otherwise.} \end{cases}$$

If ϕ is a univalent mapping of $\mathbb{D}$ onto a simply connected domain $\mathcal{D}$ then the result above implies that C_ϕ is compact if and only if $\mathcal{D}$ has no finite angular derivative, or, equivalently,

$$\lim_{|z|\to 1} \frac{1-|\phi(z)|}{1-|z|} = \infty.$$

In the next section we will define the angular derivative and other quantities pertinent to this work. In general, although the angular derivative criterion is sufficient for compactness, it is not necessary, see [8]. For an introduction to the background to these results see [7] or [2].

In [5] the author showed that if ϕ is a universal covering map onto a multiply connected domain of the form described below then the angular derivative criterion is both necessary and sufficient for C_ϕ to be compact on H^p. In particular, let $\mathcal{D} = \mathcal{D}_0 \backslash \{p_1, p_2, \ldots, p_n\}$ where $\mathcal{D}_0$ is a simply connected domain in $\mathbb{D}$ and p_i, $i = 1, \ldots, n$ are isolated points in the interior of $\mathcal{D}_0$. It was shown that if ϕ is a universal covering map of $\mathbb{D}$ onto $\mathcal{D}$ then C_ϕ is compact on H^p, $1 \le p < \infty$, if and only if

$$\lim_{|z|\to 1} \frac{1-|\phi(z)|}{1-|z|} = \infty.$$

This result highlights the importance of the geometry of the domain $\mathcal{D}$ in the characterisation of compactness of C_ϕ. In fact it was shown that if ψ is a univalent mapping of $\mathbb{D}$ onto $\mathcal{D}_0$, then C_ϕ is compact on H^p, $1 \le p < \infty$, if and only if C_ψ is.

The purpose of this paper is to extend these results to arbitrary domains of finite multiplicity.

Throughout this paper, $\mathcal{D}$ will represent a finitely connected domain contained in $\mathbb{D}$ whose boundary consists of n components that may be either points or continua. Let ϕ be the universal covering map of $\mathbb{D}$ onto $\mathcal{D}$. As in [5] we wish to characterize the compactness of $C_\phi \colon H^p \to H^p$. Our first main result is the following.

Theorem 1.1. *Suppose $\mathcal{D}$ is a finitely connected domain in $\mathbb{D}$. Let ϕ be a holomorphic universal covering map of $\mathbb{D}$ onto $\mathcal{D}$. Then C_ϕ is compact on H^p, $1 \le p < \infty$, if and only if*

$$\lim_{|z|\to 1} \frac{1-|\phi(z)|}{1-|z|} = \infty.$$

As in [5] we develop this further in the following result.

Theorem 1.2. *Suppose $\mathcal{D}$ is a finitely connected domain in $\mathbb{D}$ that can be obtained by removing finitely many components from the interior of a simply connected domain $\mathcal{D}_0$. Let ϕ be a holomorphic universal covering map of $\mathbb{D}$ onto $\mathcal{D}$ and ψ a univalent mapping of $\mathbb{D}$ onto $\mathcal{D}_0$. Then C_ϕ is compact on H^p, $1 \leq p < \infty$, if and only if C_ψ is.*

In order to prove these results we will require ideas from Fuchsian groups and Riemann surfaces. We will provide an overview of these but the reader may find more details in [1] and [3]. In Section 2 we will cover many of the prerequisites required for the proofs of the results. Sections 3 and 4 are devoted to the proofs of the main results.

2. Preliminaries

We begin this section with a discussion of the angular derivative and then move on to the construction of the universal covering map of $\mathbb{D}$ onto a multiply connected domain.

Consider a holomorphic mapping $\phi\colon \mathbb{D} \to \mathbb{D}$. At a point $\zeta \in \partial\mathbb{D}$ ϕ is said to have a finite angular derivative if there is a a $\eta \in \partial\mathbb{D}$ such that the ratio

$$\frac{\phi(z) - \eta}{z - \zeta}$$

converges as $z \to \zeta$ non-tangentially. The angular derivative, when it exists, will be denoted by $\phi'(\zeta)$. The existence of the angular derivative at a point ζ has a number of geometric consequences on the mapping properties of ϕ. For example it is known that it implies that ϕ is conformal at ζ, see also Julia's Lemma [2, Lemma 2.41].

We will require the following important result.

Theorem A (Julia-Caratheodory Theorem)**.** *Let $\phi\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function and suppose $\zeta \in \partial\mathbb{D}$. The following are equivalent.*

1. *ϕ has finite angular derivative $\phi'(\zeta)$ at ζ.*
2. $D(\zeta) = \liminf_{z\to\zeta} \dfrac{1 - |\phi(z)|}{1 - |z|} < \infty$.
3. *Both ϕ and ϕ' have non-tangential limits at ζ, with $\lim_{r\to 1} \phi(r\zeta) = \eta \in \partial\mathbb{D}$.*

When any (all) of these criteria hold we have that $D(\zeta)$ is the non-tangential limit

$$\lim_{z\to\zeta} \frac{1 - |\phi(z)|}{1 - |z|}$$

and $\phi'(\zeta) = D(\zeta)\overline{\zeta}\eta$.

In particular it is a consequence of this theorem that if

$$\lim_{|z|\to 1} \frac{1 - |\phi(z)|}{1 - |z|} = \infty$$

then ϕ cannot have finite angular derivative at any point of $\partial\mathbb{D}$.

When ϕ is univalent the existence of an angular derivative at a point depends only on the geometry of the simply connected domain $\phi(\mathbb{D})$, or, more precisely, on the boundary of $\phi(\mathbb{D})$. A good account of these results is contained in [4, §V.5]. We mention only that since the existence of an angular derivative at a point implies it is conformal there. Any simply connected domain contained in a polygon in $\mathbb{D}$, for example, cannot have finite angular derivative.

Let $\mathcal{D} \subset \mathbb{D}$ be an arbitrary multiply connected domain. Since $\mathcal{D}$ is hyperbolic, there is a Riemann surface $\mathcal{R}_{\mathbb{D}} \cong \mathbb{D}/\Gamma$ conformally equivalent to $\mathcal{D}$, where Γ is a torsion-free Fuchsian group. The universal covering map is then constructed as in the following diagram.

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\tilde{\phi}_{\mathcal{D}}} & \mathcal{R}_{\mathcal{D}} \\ & \searrow^{\phi} & \downarrow \pi \\ & & \mathcal{D} \end{array}$$

Here the mapping $\tilde{\phi}_{\mathcal{D}}$ exists as a consequence of the uniformization theorem. The mapping ϕ is conformal and locally univalent. It follows from the construction that for any $w \in \mathcal{D}$ the pre-image under ϕ of w is a Γ-orbit, $\Gamma(z) = \{g(z) \colon g \in \Gamma\}$. We will use this to estimate $\mathcal{N}_\phi(w)$ in terms of the action of Γ. As such we suppose $\mathcal{F}$ is a locally finite fundamental domain for the action of Γ on $\mathbb{D}$. Then $\tilde{\mathcal{F}}/\Gamma$ is homeomorphic to $\mathbb{D}/\Gamma$ where $\tilde{\mathcal{F}}$ denotes the relative closure of $\mathcal{F}$ in $\mathbb{D}$, see [1, §9.2]. The Dirichlet fundamental polygon is one such example, it is defined for $w \in \mathbb{D}$ as

$$D(w) = \bigcap_{g\in\Gamma, g\neq id} \{z \in \mathbb{D} \colon d_{\mathbb{D}}(z, w) < d_{\mathbb{D}}(z, g(w))\}.$$

Here we denote by $d_{\mathbb{D}}(z_1, z_2)$ the hyperbolic distance in $\mathbb{D}$,

$$d_{\mathbb{D}}(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{2}{1 - |z|^2} |dz|$$

where the infimum is taken over all smooth curves γ joining z_1 to z_2. This metric has as its geodesics radii and arcs of circles orthogonal to $\partial\mathbb{D}$. In particular

$$d_{\mathbb{D}}(0, z) = \log \frac{1 + |z|}{1 - |z|}.$$

Since automorphisms of $\mathbb{D}$ are isometries of the hyperbolic metric the action of Γ gives rise to a hyperbolic metric on $\mathbb{D}/\Gamma$. Automorphisms are characterized as parabolic or hyperbolic according to whether they have one or two fixed points on $\partial\mathbb{D}$. The limit set of Γ, denoted $\Lambda(\Gamma)$, is the set of all limit points of orbits of a point under the action of Γ, it is a proper subset of $\partial\mathbb{D}$.

The conjugacy classes of parabolic elements of Γ correspond to punctures in the Riemann surface, [3, pp 214-216]. Similarly there is a correspondence between boundary loops of $\mathcal{D}$ and conjugacy classes of hyperbolic elements, called *boundary hyperbolic elements* in [1, page 265].

As in [5] we rely on estimating the Nevanlinna counting function by the Poincare series for Γ:

$$\rho_\Gamma(z, w; s) = \sum_{g\in\Gamma} \exp -s d_{\mathbb{D}}(z, g(w)).$$

The Poincare series converges for $s > \dim \Lambda(\Gamma)$ and so, in particular, for $s = 1$, the exponent that we will require.

3. Proof of Theorem 1.1

We assume throughout that $\partial\mathcal{D} \cap \partial\mathbb{D} \neq \emptyset$. Otherwise both Theorem 1.1 and 1.2 are true trivially.

In order to utilise Shapiro's compactness criterion we must first establish that as $|w| \to 1$ in $\mathcal{D}$ any $z \in \phi^{-1}(w)$ satisfies $|z| \to 1$. Consider a locally finite fundamental domain $\mathcal{F}$ for the action of Γ on $\mathbb{D}$. Then $\mathcal{F}$ can be chosen to be a finite sided hyperbolic polygon with free sides I_k, $k = 1, 2, \ldots N$. Each free side of $\mathcal{F}$ lies in an interval of discontinuity, σ_k, for Γ on $\partial\mathbb{D}$. The stabilizer

$$\{g \in \Gamma\colon\, g(\sigma_k) = \sigma_k\}$$

is an infinite cyclic subgroup generated by a hyperbolic automorphism, h_k say.

Now the conjugacy class of each h_k, $Cl(h_k)$, maps I_k to the pairwise disjoint sets

$$J_k = \bigcup_{\tilde{h}\in Cl(h_k)} \tilde{h}(I_k) = \bigcup_{\tilde{h}\in Cl(h_k)} \tilde{h}(\sigma_k).$$

Note that the unit circle then consists of points in the limit set $\Lambda(\Gamma)$ or in J_k for some k.

Since $\mathbb{D}$ is finitely connected there is a $0 < R < 1$ such that

$$A(R, 1) \cap \partial\mathcal{D} = A(R, 1) \cap \partial\mathcal{D}_0$$

where $A(R, 1)$ denotes the annulus centered at the origin with inner radius R and outer radius 1. Therefore as $|w| \to 1$, w must converge to $\partial\mathcal{D}_0$ which, considered as a boundary loop in $\mathbb{D}/\Gamma$, implies that $z \in \phi^{-1}(w)$ converges to J_k for some k. Hence as $|w| \to 1$, $|z| \to 1$ as required. Without loss of generality, we will throughout assume J_1 corresponds to $\partial\mathcal{D}_0$.

Proposition 3.1. *With the notation above C_ϕ is compact on H^p, $1 \le p < \infty$, if and only if for each $\zeta \in J_1$*

$$\lim_{z\to\zeta} \frac{\rho_\Gamma(0, z; 1)}{1 - |\phi(z)|} = 0$$

Proof. We may write the Nevanlinna counting function as

$$\mathcal{N}_\phi(w) = \sum_{g\in\Gamma} \log \frac{1}{|g(z)|}$$

where z is an arbitrary preimage of w under ϕ.

Now, since Γ is discontinuous in $\mathbb{D}$, the set $\{g\colon |g(z)| \leq R\}$, for $1/2 < R < 1$, is finite. Therefore since

$$\log\frac{1}{x} \leq 1 - x^2 \leq 2\log\frac{1}{x}, \qquad 1/2 < x < 1$$

we have that

$$\begin{aligned} \mathcal{N}_\phi(w) &\leq C\sum_{g\in\Gamma}(1 - |g(z)|^2) \\ &\leq C\sum_{g\in\Gamma}\frac{1-|g(z)|}{1+|g(z)|} \\ &= C\sum_{g\in\Gamma}\exp -d_{\mathbb{D}}(0, g(z)) = C\rho_\Gamma(0, z; 1) \end{aligned}$$

where C denotes a constant not necessarily the same at each instance.

The opposite inequality

$$\mathcal{N}_\phi(w) \geq C\rho_\Gamma(0, z; 1)$$

follows similarly. The result now follows from the definition of J_1. □

To complete the proof of the theorem, we will require the following result that was proved in [5].

Lemma 3.2. [5, Lemma 1] *If* Γ *uniformizes the domain* $\mathcal{D}$ *then for* $z \in D(0)$ *with* z *close enough to* I_1*, the free side of* $D(0)$ *corresponding to* $\partial\mathcal{D}_0$*.*

$$c_1 \exp -d_{\mathbb{D}}(0, z) \leq \rho_\Gamma(0, z, 1) \leq c_2 \exp -d_{\mathbb{D}}(0, z)$$

where c_1 *and* c_2 *are constants depending only on* Γ*.*

Since $\exp -d(0, z) \sim (1 - |z|)$ it follows that C_ϕ is compact if and only if

$$\lim_{z\to\zeta} \frac{1-|z|}{1-|\phi(z)|} = 0$$

for any ζ in the free side of $D(0)$ corresponding to $\partial\mathcal{D}_0$. In the notation above we may call this free side I_1, then σ_1, J_1 and h_1 are implicitly defined.

To complete the proof of the theorem, we must consider, in turn, J_1, J_k $(k > 1)$ and $\Lambda(\Gamma)$.

Consider first $\zeta \in J_1$. Then there exists a $h \in \Gamma$ with $h(\zeta) \in I_1$. Now suppose without loss of generality that $z \to \zeta$ inside $D(h^{-1}(0)) = h^{-1}(D(0))$. Then, with $\zeta^* = h(\zeta)$ and $z^* = h(z)$, we have that

$$\exp -d_{\mathbb{D}}(0, z) \leq -d_{\mathbb{D}}(0, h(z)),$$

and so

$$\begin{aligned}\lim_{z\to\zeta}\frac{1-|z|}{1-|\phi(z)|} &\leq C\lim_{z\to\zeta}\frac{\exp -d_{\mathbb{D}}(0,z)}{1-|\phi(z)|}\\ &\leq C\lim_{z\to\zeta}\frac{\exp -d_{\mathbb{D}}(0,h(z))}{1-|\phi(z)|}\\ &= C\lim_{z^*\to\zeta^*}\frac{\exp -d_{\mathbb{D}}(0,z^*)}{1-|\phi(z^*)|}\\ &\leq C\lim_{z^*\to\zeta^*}\frac{\rho_\Gamma(0,z^*;1)}{1-|\phi(z^*)|}.\end{aligned}$$

Conversely,

$$\begin{aligned}\lim_{z\to\zeta}\frac{1-|z|}{1-|\phi(z)|} &= \lim_{z\to\zeta}\frac{1-|z|}{1-|z^*|}\frac{1-|z^*|}{1-|\phi(z^*)|}\\ &= \frac{1}{|h'(\zeta)|}\lim_{z^*\to\zeta^*}\frac{1-|z^*|}{1-|\phi(z^*)|}.\end{aligned}$$

Therefore

$$\lim_{z\to\zeta}\frac{1-|z|}{1-|\phi(z)|}=0 \quad \text{if and only if} \quad \lim_{z^*\to\zeta^*}\frac{\rho_\Gamma(0,z^*;1)}{1-|\phi(z^*)|}=0$$

for any $\zeta\in J_1$.

Now suppose that $\zeta\in J_k$, $k>1$. By the comments at the beginning of this section, the sets J_k correspond in the Riemann surface structure $\mathbb{D}/\Gamma$ to boundary loops corresponding to continua interior to $\mathcal{D}_0$. In particular, as $z\to\zeta\in J_k$, we have that $\phi(z)$ is contained in a compact set interior to $\mathbb{D}$. It follows that at these points ϕ cannot have a finite angular derivative by definition.

Finally, we consider the limit set $\zeta\in\Lambda(\Gamma)$. As above, these points necessarily have no finite angular derivative. This follows from the following result which may be found in [1, Theorem 10.2.5].

Lemma 3.3. *Γ is finitely generated if and only if each $\zeta\in\Lambda(\Gamma)$ is either*

(1) *a fixed point for a parabolic element of Γ; or*
(2) *a point of approximation – i.e. there is a sequence g_n, $n=1,2,\ldots$, of elements of Γ such that $g_n(0)\to\zeta$ non-tangentially.*

If ζ is a fixed point for a parabolic element, then this corresponds to a puncture in the Riemann surface structure (see [3, pp 214-216]) and therefore to an isolated point in the boundary of $\mathcal{D}$ interior to $\mathbb{D}$. Hence as above this implies that ϕ cannot have a finite angular derivative at ζ.

In the second case if $\zeta=\lim_{n\to\infty}g_n(0)$, where $(g_n(0))_{n\in\mathbb{Z}}$ is a non-tangential sequence, then ϕ is constant and has absolute value less than 1 on $(g_n(0))_{n\in\mathbb{Z}}$. Therefore, by the Julia-Caratheodory theorem, ϕ cannot have finite angular derivative at ζ. This completes the proof.

4. Proof of Theorem 1.2

In order to prove this result, we will consider the function

$$\omega = \psi^{-1} \circ \phi.$$

Now ω is a universal covering map of $\mathbb{D}$ onto a multiply connected domain with the same configuration as $\mathcal{D}$ whose outer boundary is $\partial\mathbb{D}$. We claim that at each point $\zeta \in \partial\mathbb{D}$ for which

$$\lim_{z\to\zeta} |\omega(z)| = 1$$

we have that $|\omega'(\zeta)| < \infty$.

Suppose that Γ_0 is the Fuchsian group uniformizing $\omega(\mathbb{D})$, so that

$$\omega(\mathbb{D}) \cong \mathbb{D}/\Gamma_0.$$

Then, as in the previous proof, we let I be the free side of a locally finite fundamental polygon for Γ_0. Now $I \subset \sigma$, an interval of discontinuity for Γ_0, and we let J be the image of I under the conjugacy class of the stabiliser of σ.

Clearly $\zeta \in J$ if and only if $|\omega(z)| \to 1$ as $z \to \zeta$. Fix one such ζ. Then we may find a neighborhood N of ζ such that ω is univalent on $N \cap \mathbb{D}$ and continuous on $N \cap \partial\mathcal{N}$. The continuity of ω on the boundary follows, since $\omega(N \cap \mathbb{D})$ is a Jordan domain for small enough N. Therefore ω can be extended to be holomorphic in N by the reflection principle.

Indeed the same conclusion can be made by considering the Schottky double of $\omega(\mathbb{D})$, defined as

$$\Omega(\Gamma_0)/\Gamma_0, \qquad \Omega(\Gamma_0) = \mathbb{D} \cup \mathbb{D}^* \cup (\partial\mathbb{D}\backslash\Lambda(\Gamma_0))$$

where $\mathbb{D}^* = \{z\colon |z| > 1\} \cup \{\infty\}$.

Since ω can be extended to be holomorphic at ζ, it must have a derivative there.

An application of the Julia-Caratheodory theorem implies that the angular derivative coincides with the absolute value of the derivative of ω at ζ, as required.

To complete the proof, first note that

$$C_\phi = C_\psi C_\omega$$

so if C_ψ is compact then so is C_ϕ. Conversely, if C_ϕ is compact, then for each point $\zeta \in \partial\mathbb{D}$, $|\phi'(\zeta)| = \infty$ by Theorem 1.1. Suppose then that η is a point at which $|\psi'(\eta)| < \infty$. We may find $\zeta \in \partial\mathbb{D}$ such that

$$\omega(r\zeta) \to \eta \qquad (r \to 1)$$

and, furthermore, $\zeta \in J$. However, we have shown that $|\omega'(\zeta)| < \infty$, so that by the Julia-Caratheodory theorem

$$\begin{aligned}\lim_{r\to 1} \phi'(r\zeta) &= \lim_{r\to 1} \psi'(\omega(r\zeta))\omega'(r\zeta) \\ &= \lambda\psi'(\eta)|\omega'(\zeta)|\end{aligned}$$

for some $|\lambda| = 1$. It follows that ϕ has finite angular derivative at ζ, contradicting the compactness of C_ϕ. Therefore $|\psi'(\eta)| = \infty$ for all $\eta \in \partial\mathbb{D}$ and C_ψ is compact.

References

1. A. F. Beardon, *The geometry of discrete groups*, Graduate Texts in Mathematics, Springer-Verlag, 1983.
2. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, CRC Press, Boca Raton, 1995.
3. H. M. Farkas and I. Kra, *Riemann surfaces*, Graduate Texts in Mathematics, Springer-Verlag, 1980.
4. J. B. Garnett and D. E. Marshall, *Harmonic measure*, Cambridge University Press, 2005.
5. M. M. Jones, *Compact composition operators with symbol a universal covering map*, J. Funct. Anal. (2015), no. 268, 887–901.
6. J. H. Shapiro, *The essential norm of a composition operator*, Ann. Math **125** (1987), 375–404.
7. J. H. Shapiro, *Composition operators and classical function theory*, Springer-Verlag, 1993.
8. J. H. Shapiro and P. D. Taylor, *Compact, nuclear, and Hilbert-Schmidt composition operators on H^2*, Indiana Univ. Math. J **23** (1973), 471–496.

Middlesex University, Hendon, London NW4 4BT, UK
E-mail address: m.m.jones@mdx.ac.uk