

MONADIC PSEUDOCOMPLEMENTED DISTRIBUTIVE LATTICES

ISMAEL CALOMINO, ANDREW LEWIS-SMITH, AND GUSTAVO PELAITAY

ABSTRACT. In this paper, we study the variety of pseudocomplemented distributive lattices with existential and universal quantifiers, called monadic pseudocomplemented distributive lattices. We introduce the variety of monadic KAN-algebras, which turns out to be different from the class studied in [Gomez C., Marcos M., San Martín H.J.: *On the relation of negations in Nelson algebras*. Rep. Math. Logic **56** (2021), 15–56], and prove that the category of monadic pseudocomplemented distributive lattices is equivalent to the category of centered monadic KAN-algebras, extending the results given in [Calomino I., Pelaitay G.: *A new categorical equivalence for Stone algebras*. Accepted in *Mathematica Slovaca* (2025)].

1. INTRODUCTION

In [17], Halmos introduces the class of monadic Boolean algebras as an algebraic counterpart of the one-variable fragment of classical predicate logic. A monadic Boolean algebra is a structure $\langle B, \exists \rangle$, where B is a Boolean algebra and $\exists$ is a unary operator on B , called the existential quantifier, satisfying the following conditions:

- (MB1) $\exists 0 = 0$,
- (MB2) $x \leq \exists x$,
- (MB3) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,
- (MB4) $\exists(x \vee y) = \exists x \vee \exists y$.

We can define the dual operator $\forall$, called the universal quantifier, as $\forall x = (\exists x')'$, where x' denotes the Boolean negation of x . It follows that $\forall$ satisfies the following dual properties:

- (MB5) $\forall 1 = 1$,
- (MB6) $\forall x \leq x$,
- (MB7) $\forall(x \vee \forall y) = \forall x \vee \forall y$,
- (MB8) $\forall(x \wedge y) = \forall x \wedge \forall y$.

Monadic operators have attracted considerable attention and have been systematically studied across a wide range of algebraic structures. Notable examples include monadic Heyting algebras [3], monadic MV-algebras [9], monadic NM-algebras [27, 28], monadic semi-Nelson algebras [15], monadic residuated distributive lattices [29, 30], monadic quasi-modal distributive nearlattices [4], monadic $k \times j$ -rough Heyting algebras [1], and monadic distributive lattices [11, 12, 13]. In addition, quantifiers on distributive lattices have been investigated in [6]. Recently, in [10], an interesting connection has been established between the class of monadic

2010 *Mathematics Subject Classification*. Primary 03G25; Secondary 06D15, 06E75.

Key words and phrases. Monadic operator, pseudocomplemented, Kalman's functor.

Boolean algebras with an automorphism and the class $\mathbf{Df}_2$ of diagonal-free two-dimensional cylindric algebras. In a related direction, Figallo Orellano [14] has studied MV-algebras endowed with two commuting quantifiers, providing a further generalization of these algebraic frameworks.

In the monadic Boolean setting, the existential quantifier is defined in terms of the universal quantifier using Boolean negation, which plays a fundamental role in classical logic. Therefore, it is natural to study monadic operators on bounded distributive lattices, where Boolean negation is replaced by the ‘logically weaker’ (more general) operation of pseudocomplementation. A well-known generalization of Boolean algebras is the variety of distributive pseudocomplemented lattices, also known as distributive p -algebras. These structures have been extensively studied by several authors in various papers. Among the key works addressing their properties and structures, we mention [2, 8, 18, 19, 21, 23].

Following the results of [20, 6], Gómez et al. [16] introduced the notion of an existential quantifier for the class of pseudocomplemented distributive lattices and established a categorical equivalence between the class of pseudocomplemented distributive lattices equipped with an existential quantifier and the variety of Kleene algebras with intuitionistic negation (called KAN-algebras) equipped with a weak quantifier.

The aim of this paper is to present an alternative approach to the results obtained in [16] by studying the class of pseudocomplemented distributive lattices equipped with existential and universal quantifiers, each of which are not necessarily interdefinable via the pseudocomplement operation. We then introduce a class of monadic KAN-algebras, which differs from that considered in [16], and present several constructions based on Kalman’s functor and the results given in [5].

The structure of the paper is as follows. In Section 2, we review basic notions on pseudocomplemented distributive lattices and KAN-algebras that are necessary for the subsequent developments. Section 3 introduces the class of monadic pseudocomplemented distributive lattices, establishes their fundamental properties, and presents a Glivenko-type theorem in this context. In Section 4, we define the class of monadic KAN-algebras, prove several properties, and show how they relate to monadic pseudocomplemented distributive lattices via appropriate constructions. Section 5 is devoted to various constructions for monadic KAN-algebras, including extensions of Kalman’s and Monteiro’s constructions to our monadic setting, as well as a generalization of Sendlewski’s construction. Finally, we establish a categorical equivalence between the category of monadic pseudocomplemented distributive lattices and the category of centered monadic KAN-algebras.

2. PRELIMINARIES

We recall some basic notions on pseudocomplemented distributive lattices and KAN-algebras that will be used throughout the rest of the paper.

2.1. Pseudocomplemented distributive lattices. A pseudocomplemented distributive lattice is an algebra $\langle A, \vee, \wedge, *, 0, 1 \rangle$ of type $(2, 2, 1, 0, 0)$ where $\langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice, and for all $a, b \in A$, the following condition holds: $a \wedge b = 0$ if and only if $a \leq b^*$. Thus, a^* is the greatest element of A that is disjoint from a ; that is, $a^* = \max\{x \in A : a \wedge x = 0\}$. We denote the class of pseudocomplemented distributive lattices by **PDL**.

Theorem 2.1. *Let $A \in \mathbf{PDL}$. Then:*

- (1) $0^* = 1$ and $1^* = 0$,
- (2) $x \leq x^{**}$,
- (3) $x^{***} = x^*$,
- (4) $x \leq y$ implies $y^* \leq x^*$,
- (5) $(x^{**} \wedge y^{**})^* = (x \wedge y)^*$,
- (6) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$,
- (7) $(x \vee y)^* = x^* \wedge y^*$,
- (8) $(x^{**} \vee y^{**})^{**} = (x \vee y)^{**}$.

An element a of a pseudocomplemented distributive lattice A is called dense if $a^* = 0$, and the set $D(A)$ of all dense elements is a filter of A . An element a of A is called regular if $a^{**} = a$. Denote by $R(A)$ the set of all regular elements of A . We define a binary operation on $R(A)$ given by $x \sqcup y = (x \vee y)^{**}$.

Theorem 2.2. [2, 7] *Let $A \in \mathbf{PDL}$. Then the structure $\langle R(A), \sqcup, \wedge, *, 0, 1 \rangle$ is a Boolean algebra and the binary relation θ_G on A defined by*

$$(1) \quad (x, y) \in \theta_G \iff x^* = y^*$$

is a congruence on A . Moreover, $R(A)$ is isomorphic to A/θ_G .

Definition 2.3. [22, 16] *Let $A \in \mathbf{PDL}$.*

- *A filter F of A is called Boolean if it contains all dense elements, or equivalently, $x \vee x^* \in F$, for all $x \in A$.*
- *A congruence θ of A is called Boolean if A/θ is a Boolean algebra, or equivalently, $x \vee x^* \in [1]_\theta$, for all $x \in A$.*

Lemma 2.4. [22, 16] *Let $A \in \mathbf{PDL}$. Then:*

- (1) *If θ is a Boolean congruence of A , then $[1]_\theta$ is a Boolean filter of A .*
- (2) *If F is a Boolean filter of A , then*

$$\theta(F) = \{(x, y) \in A \times A : \exists f \in F (x \wedge f = y \wedge f)\}$$

is a Boolean congruence of A .

The assignments $\theta \mapsto [1]_\theta$ and $F \mapsto \theta(F)$ define an order isomorphism between the poset of Boolean congruences of A and the poset of Boolean filters of A .

2.2. KAN-algebras.

Definition 2.5. [16, 20] *An algebra $\langle T, \vee, \wedge, \sim, c, 0, 1 \rangle$ of type $(2, 2, 1, 0, 0, 0)$ is called a centered Kleene algebra if the reduct $\langle T, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice and $\sim$ satisfies the following conditions:*

- (1) $\sim(x \wedge y) = \sim x \vee \sim y$,
- (2) $\sim \sim x = x$,
- (3) $x \wedge \sim x \leq y \vee \sim y$,
- (4) $\sim c = c$.

Definition 2.6. [16, 25] *A Kleene algebra with intuitionistic negation, or KAN-algebra for short, is an algebra $\langle T, \vee, \wedge, \sim, \neg, 0, 1 \rangle$ of type $(2, 2, 1, 1, 0, 0)$ such that $\langle T, \vee, \wedge, \sim, 0, 1 \rangle$ is a Kleene algebra and the following conditions are satisfied:*

- (K1) $\neg(x \wedge \neg(x \wedge y)) = \neg(x \wedge \neg y)$,
- (K2) $\neg(x \vee y) = \neg x \wedge \neg y$,
- (K3) $x \wedge \sim x = x \wedge \neg x$,

$$(K4) \quad \sim x \leq \neg x,$$

$$(K5) \quad \neg(x \wedge y) = \neg((\sim \neg x) \wedge y).$$

An algebra $\langle T, \vee, \wedge, \sim, \neg, c, 0, 1 \rangle$ of type $(2, 2, 1, 1, 0, 0, 0)$ is a centered KAN-algebra, or KANc-algebra for short, if $\langle T, \vee, \wedge, \sim, \neg, 0, 1 \rangle$ is a KAN-algebra and the structure $\langle T, \vee, \wedge, \sim, c, 0, 1 \rangle$ is a centered Kleene algebra.

We denote the variety of KAN-algebras by **KAN** and the variety of centered KAN-algebras by **KANc**. In any $T \in \mathbf{KAN}$ it is verified that $\neg 1 = 0$, $\neg 0 = 1$, $\neg(x \wedge \sim x) = \neg(x \wedge \neg x) = 1$, and $x \leq y$ implies $\neg y \leq \neg x$ (for more details see [16]). We adopt the following notation:

$$(2) \quad \Diamond x := \sim \neg x$$

and

$$(3) \quad \Box x := \neg \sim x.$$

Then $\Box x = \sim \Diamond \sim x$ and $\Diamond x = \sim \Box \sim x$. The operators $\Diamond$ and $\Box$ we can describe several important properties of KAN-algebras that will be used in this paper.

Proposition 2.7. [16, 5] *Let $T \in \mathbf{KAN}$. Then:*

- (1) $\Diamond 0 = 0$ and $\Box 1 = 1$,
- (2) $\Diamond 1 = 1$ and $\Box 0 = 0$,
- (3) $\Diamond x \leq x \leq \Box x$,
- (4) $\Diamond x \leq \neg \neg x \leq \Box x$,
- (5) $\neg \Diamond x = \Box \neg x = \neg x$,
- (6) $\Diamond \Diamond x = \Diamond x$,
- (7) $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$,
- (8) $\Diamond(\Diamond x \wedge \Diamond y) = \Diamond(x \wedge y)$,
- (9) $\Box \Box x = \Box x$,
- (10) $\Box(x \wedge y) = \Box x \wedge \Box y$,
- (11) $\Box(\Box x \vee \Box y) = \Box(x \vee y)$,
- (12) $x \vee \sim x = \sim x \vee \Diamond x$,
- (13) $x \wedge \sim x = x \wedge \Box \sim x$,
- (14) $x = (\Diamond x \vee \sim x) \wedge \Box x$,
- (15) $\Box x = \Box y$ and $\Diamond x = \Diamond y$ implies $x = y$,
- (16) $\Diamond(x \wedge y) = 0$ if and only if $\Diamond x \leq \Diamond \neg y$.

Lemma 2.8. [16, 5] *Let $T \in \mathbf{KANc}$. Then:*

- (1) $\neg c = 1$,
- (2) $\Diamond c = 0$ and $\Box c = 1$,
- (3) $\Diamond(c \wedge \Box x) = 0$ and $\Box(\Diamond x \vee c) = 1$,
- (4) $x = (\Diamond x \vee c) \wedge \Box x$,
- (5) $x \vee c = \Diamond x \vee c$.

Let $T \in \mathbf{KAN}$ and consider the set $T^\Diamond := \{x \in T : \Diamond x = x\}$. We define on $T^\Diamond$ the operations $x \vee^\Diamond y := \Diamond(x \vee y)$, $x \wedge^\Diamond y := \Diamond(x \wedge y)$ and $\neg^\Diamond x := \Diamond \neg x$. It is easy to see that $x \vee^\Diamond y = x \vee y$, and if $x \in T^\Diamond$, then $\sim x = \neg x$. Hence, we have $T^\Diamond = \{x \in T : \sim x = \neg x\}$. If we consider the set $T^\Box := \{x \in T : \Box x = x\}$, then $T^\Box = \sim(T^\Diamond)$, and by the results given in [16, 5], the structure $\langle T^\Diamond, \vee, \wedge, \neg^\Diamond, 0, 1 \rangle$ is a pseudocomplemented distributive lattice.

On the other hand, in [16], the authors proved that if T is a KAN-algebra, then the binary relation $\theta \subseteq T \times T$ defined by $(x, y) \in \theta$ if and only if $\neg x = \neg y$ is an

equivalence relation on T that is compatible with the operations $\vee$, $\wedge$, and $\neg$. If we denote by $[x]_\theta$ the equivalence class of x modulo θ and $T^\theta = \{[x]_\theta : x \in T\}$ the set of all equivalence classes, then $\langle T^\theta, \vee^\theta, \wedge^\theta, \neg^\theta, [0]_\theta, [1]_\theta \rangle$ is a pseudocomplemented distributive lattice, where the operations on T^θ are defined by $[x]_\theta \vee^\theta [y]_\theta := [x \vee y]_\theta$, $[x]_\theta \wedge^\theta [y]_\theta := [x \wedge y]_\theta$ and $\neg^\theta [x]_\theta := [\neg x]_\theta$. So, the order $\leq$ in T^θ is given by $[x]_\theta \leq [y]_\theta$ if and only if $\neg y \leq \neg x$. Moreover, the pseudocomplemented distributive lattices $\langle T^\diamond, \vee, \wedge^\diamond, \neg^\diamond, 0, 1 \rangle$ and $\langle T^\theta, \vee^\theta, \wedge^\theta, \neg^\theta, [0]_\theta, [1]_\theta \rangle$ are isomorphic [5].

3. MONADIC PSEUDOCOMPLEMENTED DISTRIBUTIVE LATTICES

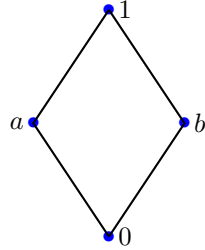
In this section, we will introduce the class of monadic pseudocomplemented distributive lattices and obtain some of their properties.

Definition 3.1. A monadic pseudocomplemented distributive lattice is a structure $\langle A, \exists, \forall \rangle$, where A is a pseudocomplemented distributive lattice and $\exists$ and $\forall$ are unary operators on A verifying (MB1) to (MB8) and the following identities:

- (MP9) $\forall \exists x = \exists x$,
- (MP10) $\exists \forall x = \forall x$,
- (MP11) $(\forall x)^* \leq \exists x^*$,
- (MP12) $(\exists x)^* \leq \forall x^*$,
- (MP13) $\forall(x \vee y) \leq \exists x \vee \forall y$,
- (MP14) $\exists x \wedge \forall y \leq \exists(x \wedge y)$.

Denote the variety of monadic pseudocomplemented distributive lattices by **MPDL**.

Example 3.2. Consider the four-element diamond $A = \{0, a, b, 1\}$, whose Hasse diagram and pseudocomplementation table are as follows:



x	x^*
0	1
a	b
b	a
1	0

The monadic operators $\exists$ and $\forall$ are defined by:

x	$\exists x$	$\forall x$
0	0	0
a	1	0
b	1	0
1	1	1

Thus, $\langle A, \exists, \forall \rangle$ is a monadic pseudocomplemented distributive lattice.

In the following lemma we obtain some properties of monadic operators.

Lemma 3.3. Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$. Then:

- (1) $\exists 1 = 1$ and $\forall 0 = 0$,
- (2) $x \leq y$ implies $\forall x \leq \forall y$ and $\exists x \leq \exists y$,
- (3) $\exists \exists x = \exists x$ and $\forall \forall x = \forall x$,
- (4) $x \leq \exists y$ if and only if $\exists x \leq \exists y$,

- (5) $\forall x \leq y$ if and only if $\forall x \leq \forall y$,
- (6) $\exists x = x$ if and only if $\forall x = x$,
- (7) $\exists x \leq y$ if and only if $x \leq \forall y$,
- (8) $\forall x = 1$ if and only if $x = 1$,
- (9) $\exists x = 0$ if and only if $x = 0$.

Proof. It is routine. $\square$

Remark 3.4. Let P and Q be two posets. A pair (f, g) of maps $f: P \rightarrow Q$ and $g: Q \rightarrow P$ forms an adjunction (or Galois connection) between P and Q if, for every $p \in P$ and $q \in Q$, we have $f(p) \leq q$ if and only if $p \leq g(q)$. By item (7) of Lemma 3.3, if $A \in \mathbf{MPDL}$, then the pair $(\exists, \forall)$ forms a Galois connection on A .

The following lemma illustrates the relationship between the monadic operators and pseudocomplementation in a pseudocomplemented distributive lattice.

Lemma 3.5. Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$. Then:

- (1) $\forall x^* = (\exists x)^*$,
- (2) $\exists x^* = (\forall x)^*$,
- (3) $(\forall x)^{**} = \forall x^{**}$,
- (4) $(\exists x)^{**} = \exists x^{**}$,
- (5) $\exists x \leq (\forall x^*)^*$,
- (6) $\forall x \leq (\exists x^*)^*$,
- (7) $\forall(\exists x \vee y) = \exists x \vee \forall y$,
- (8) $\exists(\forall x \wedge y) = \forall x \wedge \exists y$.

Proof. We will prove only (1), (5) and (7).

(1): By (MP14) and (MB1) we have $\exists x \wedge \forall x^* \leq \exists(x \wedge x^*) = \exists 0 = 0$. So, $\forall x^* \wedge \exists x = 0$ and $\forall x^* \leq (\exists x)^*$. Then, by (MP12) we have $\forall x^* = (\exists x)^*$.

(5): By (MP14) and (MB1), it follows $\exists x \wedge \forall x^* \leq \exists(x \wedge x^*) = 0$, i.e., $\exists x \leq (\forall x^*)^*$.

(7): By (MP9), (MB7) and Lemma 3.3 we have

$$\forall(\exists x \vee y) = \forall(\forall \exists x \vee y) = \forall \forall \exists x \vee \forall y = \exists x \vee \forall y.$$

Therefore, (7) is proved. $\square$

Remark 3.6. Consider the three-element chain $A = \{0, a, 1\}$, whose Hasse diagram and pseudocomplementation table are as follows:



x	x^*
0	1
a	0
1	0

The monadic operators $\exists$ and $\forall$ are defined by:

x	$\exists x$	$\forall x$
0	0	0
a	1	a
1	1	1

Thus, $\langle A, \exists, \forall \rangle$ is a monadic pseudocomplemented distributive lattice, and

$$(\exists a^*)^* = 1 \not\leq a = \forall a.$$

Therefore, unlike in the Boolean case, in the class of monadic pseudocomplemented distributive lattices the monadic operators $\exists$ and $\forall$ are not interdefinable by means of pseudocomplementation.

Lemma 3.7. *Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$ and define $M := \{x \in A : \exists x = x\}$. Then:*

- (1) $M = \{x \in A : \forall x = x\}$.
- (2) M is a subalgebra of $\langle A, \exists, \forall \rangle$.
- (3) $\exists x = \min\{y \in M : x \leq y\}$ and $\forall x = \max\{y \in M : y \leq x\}$.

Proof. (1): This follows from Lemma 3.3.

(2): By (MB4), (MB8), (MP9), and (MP10), the set M is closed under $\vee$, $\wedge$, $\forall$, and $\exists$, respectively. Since $\exists 0 = 0$ and $\exists 1 = 1$, it follows that $0, 1 \in M$. Finally, by Lemma 3.3, if $x \in M$, then $x^* = (\exists x)^* = \forall x^*$ and $x^* = (\forall x)^* = \exists x^*$. Thus, M is closed under $*$ and hence M is a subalgebra of $\langle A, \forall, \exists \rangle$.

(3): We only prove $\exists x = \min\{y \in M : x \leq y\}$. Since $\exists x = \forall \exists x$ and $x \leq \exists x$, then $\exists x \in M$ and $\min\{y \in M : x \leq y\} \leq \exists x$. On the other hand, if $y \in M$ such that $x \leq y$, then $\exists x \leq \exists y = y$ and $\exists x \leq \min\{y \in M : x \leq y\}$. $\square$

3.1. Glivenko's theorem. We prove a Glivenko style theorem for monadic pseudocomplemented distributive lattices, as a generalization of Theorem 2.2.

If $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$, then we define the operator

$$(4) \quad \exists_r x := (\exists x)^{**}.$$

By Lemma 3.5 we have $\exists_r x = \exists x^{**}$. The following result extends the Theorem 2.2.

Theorem 3.8. *Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$. Then $\langle R(A), \exists_r \rangle$ is a monadic Boolean algebra and the binary relation θ_G given by 1 is a monadic Boolean congruence on A . Moreover, the monadic Boolean algebras $R(A)$ and A/θ_G are isomorphic.*

Proof. It is clear that $\exists_r$ is well defined over $R(A)$. Since $\exists_r x = \exists x$ for all $x \in R(A)$, then it is easy to prove (MB1), (MB2) and (MB3). We see (MB4). Let $x, y \in R(A)$. Then by Theorem 2.1 and Lemma 3.5,

$$\begin{aligned} \exists_r(x \sqcup y) &= (\exists(x \vee y)^{**})^{**} = (\exists(x \vee y))^{**} \\ &= (\exists x \vee \exists y)^{**} = ((\exists x)^{**} \vee (\exists y)^{**})^{**} \\ &= (\exists_r x \vee \exists_r y)^{**} = \exists_r x \sqcup \exists_r y. \end{aligned}$$

Therefore, the structure $\langle R(A), \exists_r \rangle$ is a monadic Boolean algebra. Now, we prove that θ_G is compatible with the operators $\exists$ and $\forall$. If $(x, y) \in \theta_G$, then $x^* = y^*$. So, by Lemma 3.5, it follows $(\exists x)^* = \forall x^* = \forall y^* = (\exists y)^*$ and $(\exists x, \exists y) \in \theta_G$. Similarly, $(\forall x, \forall y) \in \theta_G$ and θ_G is a congruence on A . Finally, it follows that the monadic Boolean algebras $R(A)$ and A/θ_G are isomorphic by the homomorphism $\varphi: A \rightarrow R(A)$ given by $\varphi(x) = x^{**}$. $\square$

3.2. Monadic Boolean filters and monadic Boolean congruences. We introduce the notions of monadic Boolean filters and monadic Boolean congruences, and establish a correspondence between them in the context of monadic pseudocomplemented distributive lattices.

Definition 3.9. *Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$.*

- *A Boolean filter F of A is called monadic if it satisfies the following condition: $x \in F$ implies $\forall x \in F$.*
- *A Boolean congruence θ of A is called monadic if A/θ is a monadic Boolean algebra.*

Remark 3.10. Note that if F is a monadic Boolean filter of a monadic pseudo-complemented distributive lattice $\langle A, \exists, \forall \rangle$ and $x \in F$, then $\exists x \in F$.

Remark 3.11. Consider Example 3.2. Then the sets $F_a = \{a, 1\}$ and $F_b = \{b, 1\}$ are Boolean filters that are closed under $\exists$, but not under $\forall$.

Lemma 3.12. Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$. Then:

- (1) If θ is a monadic Boolean congruence of $\langle A, \exists, \forall \rangle$, then $[1]_\theta$ is a monadic Boolean filter of $\langle A, \exists, \forall \rangle$.
- (2) If F is a monadic Boolean filter of $\langle A, \exists, \forall \rangle$, then $\theta(F)$ is a monadic Boolean congruence of $\langle A, \exists, \forall \rangle$.

Moreover, the assignments $\theta \mapsto [1]_\theta$ and $F \mapsto \theta(F)$ define an order isomorphism between the poset of monadic Boolean congruences of $\langle A, \exists, \forall \rangle$ and the poset of monadic Boolean filters of $\langle A, \exists, \forall \rangle$.

Proof. (1) It is an immediate consequence of (MB5).

(2) Let F be a monadic Boolean filter of $\langle A, \exists, \forall \rangle$. Then $\theta(F)$ is a Boolean congruence. We prove that $\theta(F)$ is compatible with $\forall$. If $(x, y) \in \theta(F)$, then there exists $f \in F$ such that $x \wedge f = y \wedge f$. So, by (MB8), $\forall x \wedge \forall f = \forall y \wedge \forall f$ and since F is monadic filter, $\forall f \in F$. Then $(\forall x, \forall y) \in \theta(F)$.

Now, we prove that $\theta(F)$ is compatible with $\exists$. If $(x, y) \in \theta(F)$, then $(x^*, y^*) \in \theta(F)$ and by Lemma 3.5, $(\forall x^*, \forall y^*) = ((\exists x)^*, (\exists y)^*) \in \theta(F)$, i.e., there exists $f \in F$ such that $(\exists x)^* \wedge f = (\exists y)^* \wedge f$. Since F is a Boolean filter, we have $\exists x \vee (\exists x)^* \in F$ and $\exists y \vee (\exists y)^* \in F$. We take $g = (\exists x \vee (\exists x)^*) \wedge (\exists y \vee (\exists y)^*) \wedge f \in F$. Then

$$\begin{aligned} \exists x \wedge g &= \exists x \wedge (\exists x \vee (\exists x)^*) \wedge (\exists y \vee (\exists y)^*) \wedge f \\ &= \exists x \wedge (\exists y \vee (\exists y)^*) \wedge f \\ &= (\exists x \wedge \exists y \wedge f) \vee (\exists x \wedge (\exists y)^* \wedge f) \\ &= (\exists x \wedge \exists y \wedge f) \vee (\exists x \wedge (\exists x)^* \wedge f) \\ &= \exists x \wedge \exists y \wedge f. \end{aligned}$$

Analogously, we can prove $\exists y \wedge g = \exists x \wedge \exists y \wedge f$. Therefore, $(\exists x, \exists y) \in \theta(F)$ and $\theta(F)$ is a monadic Boolean congruence. $\square$

Remark 3.13. Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$. Since $D(A)$ is a monadic Boolean filter, it follows by Lemma 3.12 that $\theta(D(A))$ is a monadic Boolean congruence. Moreover, it can be shown that this congruence coincides with the Glivenko congruence θ_G defined in 1.

4. MONADIC KAN-ALGEBRAS

The aim of this section is to introduce the class of monadic KAN-algebras and establish some of their properties.

For any $T \in \mathbf{KAN}$ and unary operator $\exists: T \rightarrow T$ we define the operator

$$(5) \quad \forall x := \sim \exists \sim x.$$

It is easy to verify that $\exists x = \sim \forall \sim x$.

Definition 4.1. A monadic KAN-algebra is a structure $\langle T, \exists \rangle$, where T is a KAN-algebra and $\exists$ is a unary operator on T verifying the following identities:

- (MK1) $\exists 0 = 0$,
- (MK2) $x \leq \exists x$,
- (MK3) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,

- (MK4) $\exists(x \vee y) = \exists x \vee \exists y$,
- (MK5) $\forall \exists x = \exists x$,
- (MK6) $\forall x \wedge \exists y \leq \exists(x \wedge y)$,
- (MK7) $\forall \neg x = \neg \exists x$,
- (MK8) $\neg \forall x = \exists \neg x$.

Furthermore, if T is a KAN-algebra with center c and $\exists c = c$, we refer to $\langle T, \exists, c \rangle$ as a centered monadic KAN-algebra.

We denote the variety of monadic KAN-algebras by **MKAN** and the variety of centered monadic KAN-algebras by **MKANc**.

Lemma 4.2. *Let $\langle T, \exists \rangle \in \mathbf{MKAN}$. Then:*

- (1) $\exists 1 = 1$,
- (2) $\exists \exists x = \exists x$,
- (3) $\forall 1 = 1$ and $\forall 0 = 0$,
- (4) $\forall x \leq x$ and $\forall \forall x = \forall x$,
- (5) $\forall(x \vee \forall y) = \forall x \vee \forall y$,
- (6) $\forall(x \wedge y) = \forall x \wedge \forall y$,
- (7) $\exists \forall x = \forall x$,
- (8) $\forall \sim x \leq \neg \exists x$,
- (9) $\exists \sim x \leq \neg \forall x$,
- (10) $\forall(x \vee y) \leq \exists x \vee \forall y$.

Proof. We only prove (5), (7) and (10).

(5): By (MK3),

$$\begin{aligned} \forall(x \vee \forall y) &= \sim \exists \sim (x \vee \sim \exists \sim y) = \sim \exists(\sim x \wedge \exists \sim y) \\ &= \sim \exists \sim x \vee \sim \exists \sim y = \forall x \vee \forall y. \end{aligned}$$

(7): By (MK5) it follows that

$$\begin{aligned} \exists \forall x &= \exists \sim \exists \sim x = \sim \forall \exists \sim x \\ &= \sim \exists \sim x = \forall x. \end{aligned}$$

(10): By (MK6) we have $\forall \sim x \wedge \exists \sim y \leq \exists(\sim x \wedge \sim y) = \exists \sim (x \vee y)$. Then, $\sim \exists \sim (x \vee y) \leq \sim (\forall \sim x \wedge \exists \sim y) = \sim \forall \sim x \vee \sim \exists \sim y$. So, $\forall(x \vee y) \leq \exists x \vee \forall y$. $\square$

The operators $\Diamond$ and $\Box$ can be viewed as modal operators on KAN-algebras. The following lemma highlights the connection between these modal operators and the monadic operators.

Lemma 4.3. *Let $\langle T, \exists \rangle \in \mathbf{MKAN}$. Then:*

- (1) $\Diamond \exists x = \exists \Diamond x$ and $\Box \exists x = \exists \Box x$.
- (2) $\Diamond \forall x = \forall \Diamond x$ and $\Box \forall x = \forall \Box x$.

Proof. (1): By (2), (5) and (MK7) we have

$$\Diamond \exists x = \sim \neg \exists x = \sim \forall \neg x = \exists \sim \neg x = \exists \Diamond x.$$

Similarly, by (2), (5) and (MK8) we have

$$\Box \exists x = \neg \sim \exists x = \neg \forall \sim x = \exists \neg \sim x = \exists \Box x.$$

(2): By (2), (5) and (MK8) we have

$$\Diamond \forall x = \sim \neg \forall x = \sim \exists \neg x = \sim \exists \sim \Diamond x = \forall \Diamond x.$$

Analogously, by (3), (5) and (MK7) it follows

$$\Box \forall x = \neg \sim \forall \sim \sim x = \neg \exists \sim x = \forall \neg \sim x = \forall \Box x$$

and $\Box \forall x = \forall \Box x$, as desired. $\square$

The following result is a consequence of Lemmas 2.8, 4.2, and 4.3.

Lemma 4.4. *Let $\langle T, \exists, c \rangle \in \mathbf{MKANc}$. Then:*

- (1) $\forall c = c$,
- (2) $\Diamond \exists c = 0$ and $\Box \exists c = 1$,
- (3) $\Diamond \forall c = 0$ and $\Box \forall c = 1$,
- (4) $\forall(x \vee c) = \forall x \vee c$ and $\forall(x \wedge c) = \forall x \wedge c$,
- (5) $\exists(x \vee c) = \exists x \vee c$ and $\exists(x \wedge c) = \exists x \wedge c$.

Let $\langle T, \exists \rangle \in \mathbf{MKAN}$. It is known that $\langle T^\Diamond, \vee, \wedge^\Diamond, \neg^\Diamond, 0, 1 \rangle$ is a pseudocomplemented distributive lattice (see [5, 16]). Consider the following operators on $T^\Diamond$:

- $\exists^\Diamond x := \Diamond \exists x$,
- $\forall^\Diamond x := \Diamond \forall x$.

If $x \in T^\Diamond$, then $\exists^\Diamond x = \exists x$ and $\forall^\Diamond x = \forall x$. This leads to the following result.

Theorem 4.5. *Let $\langle T, \exists \rangle \in \mathbf{MKAN}$. Then $\langle T^\Diamond, \exists^\Diamond, \forall^\Diamond \rangle$ is a monadic pseudocomplemented distributive lattice.*

Proof. By the results developed in [16, 5] we know that $T^\Diamond$ is a pseudocomplemented distributive lattice. By Lemma 4.3 it follows that the operators $\exists^\Diamond$ and $\forall^\Diamond$ are well defined. Since $\Diamond \exists x = \exists x$ and $\Diamond \forall x = \forall x$ for all $x \in T^\Diamond$, we only prove (MB3) and (MP14). Let $x, y \in T^\Diamond$. Then:

(MB3): By (MB3), Proposition 2.7 and Lemma 4.3 we have

$$\exists^\Diamond(x \wedge^\Diamond \exists^\Diamond y) = \exists \Diamond(x \wedge \exists y) = \Diamond(\exists x \wedge \exists y) = \exists^\Diamond x \wedge^\Diamond \exists^\Diamond y.$$

(MP14): By (MP14), Proposition 2.7 and Lemma 4.3 it follows that

$$\exists^\Diamond x \wedge^\Diamond \forall^\Diamond y = \Diamond(\exists x \wedge \forall y) \leq \Diamond \exists(x \wedge y) = \exists \Diamond(\Diamond x \wedge \Diamond y) = \exists^\Diamond(x \wedge^\Diamond y).$$

The rest of the points are easily followed. $\square$

Proposition 4.6. *Let $\langle T_1, \exists_1 \rangle, \langle T_2, \exists_2 \rangle \in \mathbf{MKAN}$. If $f: T_1 \rightarrow T_2$ is a homomorphism in $\mathbf{MKAN}$, then $f^\Diamond: T_1^\Diamond \rightarrow T_2^\Diamond$ defined by $f^\Diamond(x) = \Diamond f(x)$ is a homomorphism in $\mathbf{MPDL}$.*

Proof. We show that $f^\Diamond$ is compatible with $\exists$. If $x \in T^\Diamond$, then by Lemma 4.3 and Proposition 2.7 we have

$$\exists^\Diamond f^\Diamond(x) = \exists \Diamond \Diamond f(x) = \Diamond \exists f(x) = \Diamond f(\exists x) = f^\Diamond(\exists^\Diamond x).$$

Analogously, $\forall^\Diamond f^\Diamond(x) = f^\Diamond(\forall^\Diamond x)$ and $f^\Diamond$ is a homomorphism in $\mathbf{MPDL}$. $\square$

If $\langle T, \exists \rangle \in \mathbf{MKAN}$, then the structure $\langle T^\theta, \vee^\theta, \wedge^\theta, \neg^\theta, [0]_\theta, [1]_\theta \rangle$ is a pseudocomplemented distributive lattice (see [5, 16]). We define on T^θ the following operators:

- $\exists^\theta[x]_\theta := [\exists x]_\theta$,
- $\forall^\theta[x]_\theta := [\forall x]_\theta$.

Theorem 4.7. *Let $\langle T, \exists \rangle \in \mathbf{MKAN}$. Then $\langle T^\theta, \exists^\theta, \forall^\theta \rangle$ is a monadic pseudocomplemented distributive lattice. Moreover, $\langle T^\theta, \exists^\theta, \forall^\theta \rangle$ and $\langle T^\Diamond, \exists^\Diamond, \forall^\Diamond \rangle$ are isomorphic.*

Proof. It is easy to prove that T^θ is a monadic pseudocomplemented distributive lattice. We know that the map $\varphi: T^\theta \rightarrow T^\diamond$ given by $\varphi([x]_\theta) := \diamond x$ is an isomorphism between the pseudocomplemented distributive lattices T^θ and $T^\diamond$ [5]. We see that φ is compatible with the monadic operators. By Lemma 4.3 and Proposition 2.7 we have

$$\varphi(\exists^\theta [x]_\theta) = \varphi([\exists x]_\theta) = \diamond \exists x = \diamond \exists \diamond x = \exists^\diamond \varphi([x]_\theta).$$

In an analogous way it is proved that $\varphi(\forall^\theta [x]_\theta) = \forall^\diamond \varphi([x]_\theta)$, as desired. $\square$

If $f: T_1 \rightarrow T_2$ is a morphism between two monadic KAN-algebras $\langle T_1, \exists_1 \rangle$ and $\langle T_2, \exists_2 \rangle$, then the map $f^\theta: T_1^\theta \rightarrow T_2^\theta$ defined by $f^\theta([x]_\theta) = [f(x)]_\theta$ is a morphism between monadic pseudocomplemented distributive lattices. Therefore, θ is a functor from **MKAN** to **MPDL**.

Let $\langle T, \exists, c \rangle \in \mathbf{MKANc}$. Consider $C(T) := \{x \in T : x \geq c\}$. Then the structure $\langle C(T), \vee, \wedge, \neg^c, c, 1 \rangle$, where $\neg^c x := \neg x \vee c$, is a pseudocomplemented distributive lattice (see [5]). We have the following result.

Theorem 4.8. *Let $\langle T, \exists, c \rangle \in \mathbf{MKANc}$. Then $\langle C(T), \exists, \forall \rangle$ is a monadic pseudocomplemented distributive lattice. Moreover, $\langle C(T), \exists, \forall \rangle$ and $\langle T^\diamond, \exists^\diamond, \forall^\diamond \rangle$ are isomorphic.*

Proof. By Lemma 4.4 we have $\exists x = \exists x \vee c$ and $\forall x = \forall x \vee c$ for all $x \in C(T)$. Thus, it is easy to prove (MB1)-(MB8) and (MP9)-(MP14).

Let $h: T^\diamond \rightarrow C(T)$ be the map defined by $h(x) := x \vee c$. We will show that h commutes with the monadic operators. If $x \in T^\diamond$, then by Lemma 4.4, we have

$$h(\exists^\diamond x) = \diamond \exists x \vee c = \exists(x \vee c) = \exists h(x).$$

Analogously, $h(\forall^\diamond x) = \forall h(x)$. So, $\langle C(T), \exists, \forall \rangle$ and $\langle T^\diamond, \exists^\diamond, \forall^\diamond \rangle$ are isomorphic. $\square$

Let $\langle T, \exists, c \rangle, \langle S, \exists, c \rangle \in \mathbf{MKANc}$. If $f: T \rightarrow S$ is a homomorphism of centered monadic KAN-algebras, then $C(f): C(T) \rightarrow C(S)$ defined by $C(f)(x) = f(x)$ is a homomorphism of monadic pseudocomplemented distributive lattices. Therefore the assignments $T \mapsto C(T)$ and $f \mapsto C(f)$ determine a functor C from **MKANc** to **MPDL**.

5. CONSTRUCTIONS FOR MONADIC KAN-ALGEBRAS

5.1. Kalman's construction. In [26], Kalman's construction was extended to the context of Heyting algebras, leading to a categorical equivalence between Heyting algebras and centered Nelson algebras. This construction was further generalized in [16] to the setting of pseudocomplemented distributive lattices and centered KAN-algebras. In this subsection, we extend these results to the monadic setting.

Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$. Consider the set

$$K(A) := \{(x, y) \in A \times A : x \wedge y = 0\}$$

with the following operations on $K(A)$:

$$\begin{aligned}
(a, b) \vee (d, e) &:= (a \vee d, b \wedge e), \\
(a, b) \wedge (d, e) &:= (a \wedge d, b \vee e), \\
\sim(a, b) &:= (b, a), \\
\neg(a, b) &:= (a^*, a), \\
0_K &:= (0, 1), \\
1_K &:= (1, 0), \\
c_K &:= (0, 0).
\end{aligned}$$

By the results developed in [16] we have that $\langle K(A), \vee, \wedge, \sim, \neg, c_K, 0_K, 1_K \rangle$ is a centered KAN-algebra. So, the order on $K(A)$ is given by $(a, b) \leq (d, e)$ if and only if $a \leq d$ and $e \leq b$. We define on $K(A)$ the following operators:

$$\exists_K(a, b) = (\exists a, \forall b) \text{ and } \forall_K(a, b) = (\forall a, \exists b).$$

Then we have $\forall_K(a, b) = \sim \exists_K \sim (a, b)$. Indeed,

$$\begin{aligned}
\forall_K(a, b) &= (\forall a, \exists b) \\
&= \sim (\exists b, \forall a) \\
&= \sim \exists_K(b, a) \\
&= \sim \exists_K \sim (a, b).
\end{aligned}$$

We have the following result.

Theorem 5.1. *Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$. Then $\langle K(A), \exists_K \rangle \in \mathbf{MKANc}$.*

Proof. Let $(a, b), (d, e) \in K(A)$. First we see that $\exists_K$ is well-defined. Since $(a, b) \in K(A)$ we have $a \wedge b = 0$. Then by (MP14) it follows that $\exists a \wedge \forall b \leq \exists(a \wedge b) = \exists 0 = 0$, i.e., $\exists_K(a, b) \in K(A)$. Analogously, we have $\forall_K(a, b) \in K(A)$.

(MK1): By (MB1) and (MB5) we have $\exists_K(0, 1) = (\exists 0, \forall 1) = (0, 1)$.

(MK2): Since $a \leq \exists a$ and $\forall b \leq b$ by (MB2) and (MB6), respectively, we have $(a, b) \leq (\exists a, \forall b) = \exists_K(a, b)$.

(MK3): By (MB3) and (MB7) it follows

$$\begin{aligned}
\exists_K((a, b) \wedge \exists_K(d, e)) &= \exists_K(a \wedge \exists d, b \vee \forall e) \\
&= (\exists a \wedge \exists d, \forall b \vee \forall e) \\
&= (\exists a, \forall b) \wedge (\exists d, \forall e) \\
&= \exists_K(a, b) \wedge \exists_K(d, e).
\end{aligned}$$

(MK4): We have by (MB4) and (MB8) that

$$\begin{aligned}
\exists_K((a, b) \vee (d, e)) &= (\exists(a \vee d), \forall(b \wedge e)) \\
&= (\exists a \vee \exists d, \forall b \wedge \forall e) \\
&= (\exists a, \forall b) \vee (\exists d, \forall e) \\
&= \exists_K(a, b) \vee \exists_K(d, e).
\end{aligned}$$

(MK5): By (MP9) and (MP10), $\forall_K \exists_K(a, b) = (\forall \exists a, \exists \forall b) = (\exists a, \forall b) = \exists_K(a, b)$.

(MK6): By (MP13) and (MP14) it follows

$$\begin{aligned}
\forall_K(a, b) \wedge \exists_K(d, e) &= (\forall a \wedge \exists d, \exists b \vee \forall e) \\
&\leq (\exists(a \wedge d), \forall(b \vee e)) \\
&= \exists_K(a \wedge d, b \vee e) \\
&= \exists_K((a, b) \wedge (d, e)).
\end{aligned}$$

(MK7): By Lemma 3.5 (1) we have

$$\begin{aligned}\forall_K \neg(a, b) &= (\forall a^*, \exists a) \\ &= ((\exists a)^*, \exists a) \\ &= \neg(\exists a, \forall b) \\ &= \neg \exists_K(a, b).\end{aligned}$$

(MK8): By Lemma 3.5 (2) it follows that

$$\begin{aligned}\neg \forall_K(a, b) &= ((\forall a)^*, \forall a) \\ &= (\exists a^*, \forall a) \\ &= \exists_K(a^*, a) \\ &= \exists_K \neg(a, b).\end{aligned}$$

Finally, by (MB1) and Lemma 3.3 (1), we have $\exists_K(0, 0) = (\exists 0, \forall 0) = (0, 0)$ and the structure $\langle K(A), \exists_K \rangle$ is a centered monadic KAN-algebra. $\square$

Remark 5.2. Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$. Since $\langle K(A), \exists_K \rangle$ is a centered monadic KAN-algebra, then we note that if $(a, b) \in K(A)$ we have

$$\diamond(a, b) = \sim \neg(a, b) = \sim (a^*, a) = (a, a^*)$$

and

$$\square(a, b) = \neg \sim (a, b) = \neg(b, a) = (b^*, b).$$

Then $K(A)^\diamond = \{(a, a^*) : a \in A\}$ and $K(A)^\square = \{(b^*, b) : b \in A\}$.

We denote by $\mathbf{MPDL}$ the category whose objects are monadic pseudocomplemented distributive lattices and by $\mathbf{MKANc}$ the category whose objects are centered monadic KAN-algebras. In both cases, the morphisms are the corresponding algebra homomorphisms. Furthermore, if $\langle A, \exists, \forall \rangle$ and $\langle B, \exists, \forall \rangle$ are two monadic pseudocomplemented distributive lattices and $h : A \rightarrow B$ is a morphism in $\mathbf{MPDL}$, then it is easy to see that the map $K(h) : K(A) \rightarrow K(B)$ defined by $K(h)(x, y) = (h(x), h(y))$ is a morphism in $\mathbf{MKANc}$. It is evident that these assignments establish a functor K from $\mathbf{MPDL}$ to $\mathbf{MKANc}$. Moreover, if $\langle T, \exists \rangle$ and $\langle S, \exists \rangle$ are two centered monadic KAN-algebras and $f : T \rightarrow S$ is a homomorphism of monadic centered KAN-algebras, then $C(f) : C(T) \rightarrow C(S)$ defined by $C(f)(x) = f(x)$ is a homomorphism of monadic pseudocomplemented distributive lattices. Is clear that the assignments $T \mapsto C(T)$ and $f \mapsto C(f)$ determine a functor C from $\mathbf{MKANc}$ to $\mathbf{MPDL}$.

Lemma 5.3. Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$. Then the map $\alpha : A \rightarrow C(K(A))$ defined by $\alpha(x) := (x, 0)$ is an isomorphism in $\mathbf{MPDL}$.

Proof. We only prove that α is compatible with the existential and universal quantifiers. If $x \in A$, then by Lemma 3.3 (1) and (MB1) we have

$$\alpha(\exists x) = (\exists x, 0) = (\exists x, \forall 0) = \exists_K(x, 0) = \exists_K \alpha(x)$$

and

$$\alpha(\forall x) = (\forall x, 0) = (\forall x, \exists 0) = \forall_K(x, 0) = \forall_K \alpha(x)$$

as claimed. $\square$

Lemma 5.4. Let $\langle T, \exists, c \rangle \in \mathbf{MKANc}$. Then the map $\beta : T \rightarrow K(C(T))$ defined by $\beta(x) := (x \vee c, \sim x \vee c)$ is an isomorphism in $\mathbf{MKANc}$.

Proof. We only prove that β commutes with the existential quantifier. Let $x \in T$. Then by Lemma 4.4,

$$\begin{aligned}\beta(\exists x) &= (\exists x \vee c, \sim \exists x \vee c) \\ &= (\exists x \vee c, \forall \sim x \vee c) \\ &= (\exists(x \vee c), \forall(\sim x \vee c)) \\ &= \exists_K(x \vee c, \sim x \vee c) \\ &= \exists_K \beta(x).\end{aligned}$$

Therefore, $\langle T, \exists \rangle$ and $\langle K(C(T)), \exists_K \rangle$ are isomorphic. $\square$

Remark 5.5. *The isomorphism β from Lemma 5.4 also commutes with the universal quantifier $\forall$. This follows from the fact that β commutes with $\sim$ and with $\exists$, together with equation (5).*

By the results established in this subsection, we prove the following theorem.

Theorem 5.6. *The functors K and C establish a categorical equivalence between MPDL and MKANC with natural isomorphism α and β .*

We conclude this subsection by presenting an example that highlights the motivation behind studying monadic pseudocomplemented distributive lattices.

Let $T \in \mathbf{KAN}$. In [16], the authors introduced the notion of a weak quantifier as a map $\Omega: T \rightarrow T$ satisfying the following conditions:

- (W1) $\Omega 0 = 0$,
- (W2) $\neg \Omega x \leq \neg x$,
- (W3) $\neg \Omega(x \wedge \Omega y) = \neg(\Omega x \wedge \Omega y)$,
- (W4) $\Omega(x \vee y) = \Omega x \vee \Omega y$,
- (W5) $\neg \Omega x = \sim \Omega x$,
- (W6) $\Omega x = \Omega \sim \neg x$.

For $A \in \mathbf{PDL}$ and an existential quantifier $\exists$ on A , that is, $\exists$ satisfies (MB1)-(MB4), we can define on $K(A)$ the operator $\Omega_\exists: K(A) \rightarrow K(A)$ by

$$(6) \quad \Omega_\exists(a, b) = (\exists a, (\exists a)^*).$$

It was shown in [16] that $\Omega_\exists$ is a weak quantifier on the KAN-algebra $K(A)$.

On the other hand, given $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$, since $\exists$ is an existential quantifier on A , we can construct a weak quantifier $\Omega_\exists$ on $K(A)$ as defined in (6). However, this operator does not coincide with the operator $\exists_K$ defined on $K(A)$, as illustrated by the following example.

Example 5.7. *We consider the monadic pseudocomplemented distributive lattice $\langle A, \exists, \forall \rangle$ from Remark 3.6. Applying Kalman's construction yields the centered KAN-algebra $K(A) = \{(0, 1), (0, a), (0, 0), (a, 0), (1, 0)\}$, illustrated by the Hasse diagram below, together with the associated strong and weak negations shown in the table.*

$(1, 0)$		
$(a, 0)$		
$(0, 0)$		
$(0, a)$		
$(0, 1)$		

(x, y)	$\sim (x, y)$	$\neg(x, y)$
$(1, 0)$	$(0, 1)$	$(0, 1)$
$(a, 0)$	$(0, a)$	$(0, a)$
$(0, 0)$	$(0, 0)$	$(1, 0)$
$(0, a)$	$(a, 0)$	$(1, 0)$
$(0, 1)$	$(1, 0)$	$(1, 0)$

Thus, taking into account the definitions given in 6 we obtain the operators $\Omega_{\exists}$ and $\exists_K$ as shown in the following table:

(x, y)	$\exists_K(x, y)$	$\Omega_{\exists}(x, y)$
$(1, 0)$	$(1, 0)$	$(1, 0)$
$(a, 0)$	$(1, 0)$	$(1, 0)$
$(0, 0)$	$(0, 0)$	$(0, 1)$
$(0, a)$	$(0, a)$	$(0, 1)$
$(0, 1)$	$(0, 1)$	$(0, 1)$

5.2. Monteiro's construction. Monteiro's construction shows that, given a Nelson algebra, one can obtain a centered Nelson algebra. A similar process was established for KAN-algebras in [5]. In this subsection, we extend this result to the monadic setting.

Let $\langle T, \exists \rangle \in \mathbf{MKAN}$. Consider the set

$$M(T) := \{(x, y) \in T^{\Diamond} \times T^{\Box} : x \leq y\}$$

with the following operations on $M(T)$:

$$\begin{aligned}
(a, b) \cup (d, e) &:= (a \vee d, \Box(b \vee e)), \\
(a, b) \cap (d, e) &:= (\Diamond(a \wedge d), b \wedge e), \\
\boxplus(a, b) &:= (\sim b, \sim a), \\
\otimes(a, b) &:= (\Diamond \neg a, \sim a), \\
0_M &:= (0, 0), \\
1_M &:= (1, 1), \\
c_M &:= (0, 1).
\end{aligned}$$

By the results in [5], $\langle M(T), \cup, \cap, \boxplus, \otimes, c_M, 0_M, 1_M \rangle$ is a centered KAN-algebra. The order on $M(T)$ is given by $(a, b) \subseteq (d, e)$ if and only if $a \leq d$ and $b \leq e$. We define on $M(T)$ the following operators:

$$\exists_M(a, b) = (\exists a, \exists b) \text{ and } \forall_M(a, b) = (\forall a, \forall b).$$

It is easy to prove that $\forall_M(a, b) = \boxplus \exists_M \boxplus (a, b)$.

Theorem 5.8. *Let $\langle T, \exists \rangle \in \mathbf{MKAN}$. Then $\langle M(T), \exists_M \rangle \in \mathbf{MKANc}$. Moreover, $\langle T, \exists \rangle$ is isomorphic to a subalgebra of $\langle M(T), \exists_M \rangle$.*

Proof. Let $(a, b) \in M(T)$. Then $\Diamond a = a$, $\Box b = b$ and $a \leq b$. Hence, by Lemma 4.3, we have $\Diamond \exists a = \exists a$, $\Box \exists b = \exists b$ and $\exists a \leq \exists b$. Thus, $\exists_M(a, b) \in M(T)$. Therefore, $\exists_K$

is well-defined. Now, we proceed to prove that the conditions (MK1)–(MK8) are satisfied. To this end, let $(a, b), (d, e) \in M(T)$.

(MK1): By (MB1), $\exists_M(0, 0) = (\exists 0, \exists 0) = (0, 0)$.

(MK2): By (MB2), $(a, b) \subseteq (\exists a, \exists b) = \exists_M(a, b)$.

(MK3): From Lemma 4.3 (1) and (MB3) we have

$$\begin{aligned} \exists_M((a, b) \cap \exists_M(d, e)) &= \exists_M(\Diamond(a \wedge \exists d), b \wedge \exists e) \\ &= (\exists \Diamond(a \wedge \exists d), \exists(b \wedge \exists e)) \\ &= (\Diamond(\exists a \wedge \exists d), \exists b \wedge \exists e) \\ &= (\exists a, \exists b) \cap (\exists d, \exists e) \\ &= \exists_M(a, b) \cap \exists_M(d, e). \end{aligned}$$

(MK4): By Lemma 4.3 (1) and (MB4),

$$\begin{aligned} \exists_M((a, b) \cup (d, e)) &= (\exists(a \vee d), \exists \Box(b \vee e)) \\ &= (\exists a \vee \exists d, \Box(\exists b \vee \exists e)) \\ &= (\exists a, \exists b) \cup (\exists d, \exists e) \\ &= \exists_M(a, b) \cup \exists_M(d, e). \end{aligned}$$

(MK5): By (MK5) we have $\forall_M \exists_M(a, b) = (\forall \exists a, \forall \exists b) = (\exists a, \exists b) = \exists_M(a, b)$.

(MK6): From Lemma 4.3 (1) and (MB4) it follows that

$$\begin{aligned} \forall_M(a, b) \cap \exists_M(d, e) &= (\Diamond(\forall a \wedge \exists d), \forall b \wedge \exists e) \\ &\subseteq (\Diamond \exists(a \wedge d), \exists(b \wedge e)) \\ &= (\exists \Diamond(a \wedge d), \exists(b \wedge e)) \\ &= \exists_M(\Diamond(a \wedge d), b \wedge e) \\ &= \exists_M((a, b) \cap (d, e)). \end{aligned}$$

(MK7): By Lemma 4.3 (2) and (MK7),

$$\begin{aligned} \forall_M \otimes (a, b) &= (\forall \Diamond \neg a, \forall \sim a) \\ &= (\Diamond \forall \neg a, \sim \exists a) \\ &= (\Diamond \neg \exists a, \sim \exists a) \\ &= \otimes \exists_M(a, b). \end{aligned}$$

(MK8): From Lemma 4.3 (1) and (MK8), we have

$$\begin{aligned} \otimes \forall_M(a, b) &= (\Diamond \neg \forall a, \sim \forall a) \\ &= (\Diamond \exists \neg a, \exists \sim a) \\ &= (\exists \Diamond \neg a, \exists \sim a) \\ &= \exists_M \otimes (a, b). \end{aligned}$$

From the results in [5], we know that T is isomorphic to a subalgebra of $M(T)$ via the map $\delta: T \rightarrow M(T)$ given by $\delta(x) = (\Diamond x, \Box x)$. We now show that δ is compatible with the existential quantifier. By Lemma 4.3, we have

$$\delta(\exists x) = (\Diamond \exists x, \Box \exists x) = (\exists \Diamond x, \exists \Box x) = \exists_M(\Diamond x, \Box x) = \exists_M(\delta(x)).$$

Therefore, $\langle T, \exists \rangle$ is isomorphic to a subalgebra of $\langle M(T), \exists_M \rangle$. $\square$

Let $\mathbf{MIKAN}$ denote the category of monadic KAN-algebras where morphisms are the corresponding algebra homomorphisms. Notably, if $\langle T_1, \exists \rangle$ and $\langle T_2, \exists \rangle$ are centered monadic KAN-algebras and $f: T_1 \rightarrow T_2$ is a morphism in $\mathbf{MIKAN}$, then f necessarily preserves the center. Indeed, we have $f(c) = f(\sim c) = \sim f(c)$ and thus, by the uniqueness of the center, $f(c) = c$. Therefore, $\mathbf{MIKAN}_C$ is a full subcategory of $\mathbf{MIKAN}$. Additionally, if $f: T_1 \rightarrow T_2$ is a morphism in $\mathbf{MIKAN}$, then

$M(f): M(T_1) \rightarrow M(T_2)$ defined by $M(f)(x, y) = (f(x), f(y))$ for all $(x, y) \in M(T_1)$ is a morphism in **MKANC**. By Theorem 5.8, M is a functor from the category **MIKAN** to **MKANC**.

Now, we explore the relationship between Kalman's construction and Monteiro's construction.

Theorem 5.9. *Let $\langle T, \exists \rangle \in \mathbf{MKAN}$. Then the centered monadic KAN-algebras $\langle M(T), \exists_M \rangle$ and $\langle K(T^\diamond), \exists_K^\diamond \rangle$ are isomorphic.*

Proof. By the results established in [5], $M(T)$ is isomorphic to $K(T^\diamond)$ via the map $t: M(T) \rightarrow K(T^\diamond)$ defined by $t(x, y) = (x, \sim y)$. It only remains to prove that t commutes with the existential quantifier. Let $(a, b) \in M(T)$. Then $\diamond a = a$, $\Box b = b$, and $a \leq b$. Therefore, by Lemma 4.3,

$$\begin{aligned} t(\exists_M(a, b)) &= (\exists a, \sim \exists b) \\ &= (\exists \diamond a, \sim \exists \Box b) \\ &= (\exists \diamond a, \forall \sim \Box b) \\ &= (\exists \diamond a, \forall \diamond \sim b) \\ &= (\diamond \exists a, \diamond \forall \sim b) \\ &= (\exists \diamond a, \forall \diamond \sim b) \\ &= \exists_K^\diamond t(a, b), \end{aligned}$$

i.e., $t(\exists_M(a, b)) = \exists_K^\diamond t(a, b)$. □

Corollary 5.10. *Let $\langle T, \exists \rangle \in \mathbf{MKANc}$. Then the monadic centered KAN-algebras $\langle T, \exists \rangle$ and $\langle M(T), \exists_M \rangle$ are isomorphic.*

Proof. This follows from the results in [5] and Theorem 5.9. □

5.3. Sendlewski's construction. In order to generalize the construction given by Sendlewski in [24], the authors of [16] studied the variety of KAN-algebras and their connection with pairs consisting of a pseudocomplemented distributive lattice and a Boolean filter. They showed that if A is a pseudocomplemented distributive lattice and $F \subseteq A$ is a Boolean filter, then the set

$$K(A, F) := \{(a, b) \in A \times A : a \wedge b = 0 \text{ and } a \vee b \in F\}$$

endowed with the same operations as $K(A)$, constitutes the universe of an algebra in **KAN**. Conversely, given any $T \in \mathbf{KAN}$, there exists a congruence θ with respect to the operations $\vee$, $\wedge$, and $\neg$ such that T^θ is a pseudocomplemented distributive lattice and the quotient set T^+/θ , where

$$T^+ := \{x \in T : \Box x = 1\}$$

is a Boolean filter (see [16]).

We denote by **PDLF** the category whose objects are pairs $\langle A, F \rangle$, where A is an object of the category **PDL** and F is a Boolean filter of A . The morphisms $f: \langle A, F \rangle \rightarrow \langle A', F' \rangle$ are morphisms $f: A \rightarrow A'$ in **PDL** such that $f(F) \subseteq F'$. Therefore, by the results given in [16], there is a categorical equivalence between the category **PDLF** and the category **KAN**.

Denote by **MPDLF** the category whose objects are pairs consisting of an algebra $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$ and a monadic Boolean filter F of A . The morphisms of **MPDLF** are the morphisms of **PDLF** that preserve the respective monadic operators.

Proposition 5.11. *Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$ and let $F \subseteq A$ be a monadic Boolean filter. Then $\langle K(A, F), \exists_K \rangle \in \mathbf{MKAN}$. Moreover, K extends to a functor from $\mathbf{MPDLF}$ to $\mathbf{MKAN}$.*

Proof. Let $(a, b) \in K(A, F)$. Then $a \wedge b = 0$ and $a \vee b \in F$. Since F is a monadic Boolean filter it follows $\forall(a \vee b) \in F$. By Lemma 4.2 (10) we have $\forall a \vee \exists b \in F$. Moreover, by axioms (MK6) and (MK1), we obtain $\forall a \wedge \exists b = 0$. Hence, $\exists_K(a, b) \in K(A, F)$ and therefore $\exists_K$ is well-defined. On the other hand, similarly to the proof of Theorem 5.1, it can be shown that $\exists_K$ satisfies the conditions (MK1)-(MK8). Therefore, it only remains to prove that K is functorial.

Let $f: \langle A, \exists, \forall, F \rangle \rightarrow \langle A', \exists', \forall', F' \rangle$ be a morphism in $\mathbf{MPDLF}$. For each $(a, b) \in K(A, F)$ we have

$$\begin{aligned} K(f)(\exists_K(a, b)) &= K(f)(\exists a, \forall b) \\ &= (f(\exists a), f(\forall b)) \\ &= (\exists' f(a), \forall' f(b)) \\ &= \exists'_K(f(a), f(b)) \\ &= \exists'_K(K(f)(a, b)), \end{aligned}$$

as desired. $\square$

Remark 5.12. *Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$ and let $F \subseteq A$ be a monadic Boolean filter. Then, by Theorems 4.7 and 5.1, we have $\langle K(A, F)/\theta, \exists_K^\theta, \forall_K^\theta \rangle \in \mathbf{MPDL}$, where the monadic operators $\exists_K^\theta$ and $\forall_K^\theta$ are defined by*

$$\exists_K^\theta([(a, b)]_\theta) = [(\exists a, \forall b)]_\theta \text{ and } \forall_K^\theta([(a, b)]_\theta) = [(\forall a, \exists b)]_\theta.$$

Lemma 5.13. *Let $\langle A, \exists, \forall \rangle \in \mathbf{MPDL}$ and let $F \subseteq A$ be a monadic Boolean filter. Then the map $g^{-1}: A \rightarrow K(A, F)/\theta$ defined by $g^{-1}(x) := [(x, x^*)]_\theta$ is an isomorphism.*

Proof. Taking into account the results of [16], it remains to show that g^{-1} preserves the monadic operators $\exists$ and $\forall$. Let $a \in A$. By Lemma 3.5 (1) we have

$$\begin{aligned} g^{-1}(\exists a) &= [(\exists a, (\exists a)^*)]_\theta \\ &= [(\exists a, \forall a^*)]_\theta \\ &= \exists_K^\theta([(a, a^*)]_\theta) \\ &= \exists_K^\theta g^{-1}(a), \end{aligned}$$

and by Lemma 3.5 (2),

$$\begin{aligned} g^{-1}(\forall a) &= [(\forall a, (\forall a)^*)]_\theta \\ &= [(\forall a, \exists a^*)]_\theta \\ &= \forall_K^\theta([(a, a^*)]_\theta) \\ &= \forall_K^\theta g^{-1}(a). \end{aligned}$$

Therefore, g^{-1} is an isomorphism. $\square$

Lemma 5.14. *Let $\langle T, \exists \rangle \in \mathbf{MKAN}$. Then $\langle K(T^\theta, T^+/\theta), \exists_K \rangle \in \mathbf{MKAN}$.*

Proof. First we show that T^+/θ is a monadic Boolean filter. It is clear that T^+/θ is a Boolean filter (see [16, Lemma 2.14]). To prove that it is monadic, we must show that it is closed under the operator $\forall$. Let $x \in T^+$. Then $\Box x = 1$ and by Lemmas 4.3 and 4.2 it follows that $\Box \forall x = \forall \Box x = \forall 1 = 1$. Hence, $\forall x \in T^+$ and T^+/θ is a monadic Boolean filter. By Proposition 5.11 and Theorem 4.7 the result follows. $\square$

Lemma 5.15. *Let $\langle T, \exists \rangle \in \mathbf{MKAN}$. Then the map $\rho: T \rightarrow K(T^\theta, T^+/\theta)$ defined by $\rho(x) = ([x]_\theta, [\sim x]_\theta)$ is an isomorphism.*

Proof. From the results of [16], it remains to show that ρ commutes with the operator $\exists$. Indeed,

$$\begin{aligned} \rho(\exists x) &= ([\exists x]_\theta, [\sim \exists x]_\theta) \\ &= ([\exists x]_\theta, [\forall \sim x]_\theta) \\ &= (\exists^\theta [x]_\theta, \forall^\theta [\sim x]_\theta) \\ &= \exists_K^\theta ([x]_\theta, [\sim x]_\theta) \\ &= \exists_K^\theta \rho(x), \end{aligned}$$

which completes the proof. $\square$

Remark 5.16. *If $T, T' \in \mathbf{MKAN}$ and $f: T \rightarrow T'$ is a morphism in $\mathbf{MKAN}$, then $f^\theta: T^\theta \rightarrow T'/\theta$ given by $f^\theta([x]_\theta) = [f(x)]_\theta$ is a morphism in $\mathbf{MPDLF}$.*

Theorem 5.17. *The categories $\mathbf{MPDLF}$ and $\mathbf{MKAN}$ are categorically equivalent.*

CONCLUSIONS AND FUTURE WORK

In this work, we have introduced and studied the variety of monadic pseudocomplemented distributive lattices, thereby extending the classical framework of monadic Boolean algebras to a broader non-Boolean context. We have established several fundamental properties, presented a Glivenko-type theorem, and described a categorical equivalence with classes of centered monadic KAN-algebras. Furthermore, we have generalized various classical constructions – such as those of Kalman, Monteiro, and Sendlewski – to the monadic setting, and we have clarified the relationships among these three constructions.

Future research may proceed along several directions. For example, it would be interesting to investigate the representation theory and dualities for monadic pseudocomplemented distributive lattices. Another promising line of research concerns the study of further classes of quantifiers or alternative monadic operators in related varieties, such as Stone algebras.

REFERENCES

- [1] Almiñana Reinoso F., Pelaitay G.: *Monadic $k \times j$ -rough Heyting algebras*. Arch. Math. Logic **60** (2021), 923–949. 1
- [2] Balbes R., Dwinger P.: *Distributive Lattices*. University of Missouri Press, Columbia, 1974. 1, 2.2
- [3] Bezhanishvili G.: *Varieties of monadic Heyting algebras. Part I*. Stud. Log. **61** (1998), 367–402. 1
- [4] Calomino I.: *Monadic quasi-modal distributive nearlattices*. Commentat. Math. Univ. Carol. **64** (2023), 161–174. 1
- [5] Calomino I., Pelaitay G.: *A new categorical equivalence for Stone algebras*. Accepted in Mathematica Slovaca (2025). 1, 2.7, 2.8, 2.2, 4, 4, 4, 5.2, 5.2, 5.2, 5.2
- [6] Cignoli R.: *Quantifiers on distributive lattices*. Discrete Math. **96** (1991), 183–197. 1
- [7] Chajda I., Halaš R., Kühr J.: *Semilattice Structures*. Heldermann Verlag, Research and Exposition in Mathematics, 2007. 2.2
- [8] Davey B.: *Subdirectly irreducible distributive double p -algebras*. Algebra Univers. **8** (1978), 73–88. 1
- [9] Di Nola A., Grigolia R.: *On monadic MV-algebras*. Ann. Pure Appl. Logic **128** (2004), 125–139. 1
- [10] Figallo A.V., Gomes C.M.: *Monadic Boolean algebras with an automorphism and their relation to $\mathbf{Df}_2$ -algebras*. Soft Comput. **24** (2020), 227–236. 1

- [11] Figallo A.V., Pascual I.B.: *Topological dualities for strong monadic distributive lattices and applications*. J. Mult.-Val. Log. Soft Comput. **32** (2019), 499–540. 1
- [12] Figallo A.V., Pascual I.B., Ziliani A.: *Monadic distributive lattices and monadic augmented Kripke frames*. J. Mult.-Val. Log. Soft Comput. **22** (2014), 189–216. 1
- [13] Figallo A.V., Pascual I.B., Ziliani A.: *Monadic distributive lattices*. Log. J. IGPL **15** (2007), 535–551. 1
- [14] Figallo Orellano, A.: *A preliminary study of MV-algebras with two quantifiers which commute*. Stud. Log. **104** (2016), 931–956. 1
- [15] Ghorbani S.: *On monadic semi-Nelson algebras*. J. Appl. Log.-IfCoLog J. Log. Appl. **11** (2024), 625–643. 1
- [16] Gomez C., Marcos M., San Martín H.J.: *On the relation of negations in Nelson algebras*. Rep. Math. Logic **56** (2021), 15–56. 1, 2.3, 2.4, 2.5, 2.6, 2.2, 2.7, 2.8, 2.2, 4, 4, 4, 5.1, 5.1, 5.1, 5.3, 5.3, 5.3, 5.3
- [17] Halmos P.: *Algebraic logic I. Monadic Boolean algebras*. Compos. Math. **12** (1956), 217–249. 1
- [18] Hecht T., Katriňák T.: *Principal congruences of p -algebras and double p -algebras*. Proc. Amer. Math. Soc. **58** (1976), 25–31. 1
- [19] Katriňák T.: *Free distributive p -algebras: a new approach*. Glasgow Math. J. **40** (1998), 333–342. 1
- [20] Kalman J.A.: *Lattices with involution*. Trans. Amer. Math. Soc. **87** (1958), 485–491. 1, 2.5
- [21] Lakser H.: *The structure of pseudocomplemented distributive lattices. I. Subdirect decomposition*. Trans. Amer. Math. Soc. **156** (1971), 334–342. 1
- [22] Rao M.S., Shum K.: *Boolean filters of distributive lattices*. International Journal of Math. and Soft Comput. **3** (2013), 41–48. 2.3, 2.4
- [23] Schmid J.: *Algebraically closed distributive p -algebras*. Algebra Univers. **15** (1982), 126–141. 1
- [24] Sendlewski A.: *Nelson algebras through Heyting ones. I*. Stud. Log. **49** (1990), 105–126. 5.3
- [25] Sendlewski A.: *Topologicality of Kleene algebras with a weak pseudocomplementation over distributive p -algebras*. Rep. Math. Logic **25** (1991), 13–56. 2.6
- [26] Vakarelov D.: *Notes on N -lattices and constructive logic with strong negation*. Stud. Log. **36** (1977), 109–125. 5.1
- [27] Wang J., He P., She Y.: *Monadic NM-algebras*. Log. J. IGPL **27** (2019), 812–835. 1
- [28] Wang J., He P., Yang J., Wang M., He X.: *Monadic NM-algebras: an algebraic approach to monadic predicate nilpotent minimum logic*. J. Logic Comput. **32** (2022), 741–766. 1
- [29] Wang J.T., She Y.H., He P.F., Ma N.N.: *On categorical equivalence of weak monadic residuated distributive lattices and weak monadic c -differential residuated distributive lattices*. Stud. Log. **111** (2023), 361–390. 1
- [30] Wang J., Wu H., He P., She Y.: *An algebraic proof of completeness for monadic fuzzy predicate logic MMTLV*. Rev. Symb. Log. **18** (2025), 213–239. 1

FACULTAD DE CIENCIAS Y ARTES, UNIVERSIDAD CATÓLICA DE ÁVILA, C/CANTEROS S/N, 05005 ÁVILA, ESPAÑA.

Email address: ismaelm.calomino@ucavila.es

DEPARTMENT OF COMPUTER SCIENCE, MIDDLESEX UNIVERSITY, LONDON, UK.

Email address: a.lewis-smith@mdx.ac.uk

CONICET AND INSTITUTO DE CIENCIAS BÁSICAS, UNIVERSIDAD NACIONAL DE SAN JUAN, 5400 SAN JUAN, ARGENTINA.

Email address: gpelaitay@gmail.com