QCSP on semicomplete digraphs

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Abstract. We study the (non-uniform) quantified constraint satisfaction problem QCSP(H) as H ranges over semicomplete digraphs. We obtain a complexity-theoretic trichotomy: QCSP(H) is either in P, is NP-complete or is Pspace-complete. The largest part of our work is the algebraic classification of precisely which semicompletes enjoy only essentially unary polymorphisms, which is combinatorially interesting in its own right.

1 Introduction

The *quantified constraint satisfaction problem* QCSP(B), for a fixed *template* (structure) B, is a popular generalisation of the *constraint satisfaction problem* CSP(B). In the latter, one asks if a primitive positive sentence (the existential quantification of a conjunction of atoms) Φ is true on B, while in the former this sentence may be positive Horn (where universal quantification is also permitted). Much of the theoretical research into CSPs is in respect of a large complexity classification project – it is conjectured that CSP(B) is always either in P or NP-complete [11]. This *dichotomy* conjecture remains unsettled, although dichotomy is now known on substantial classes (e.g. structures of size ≤ 3 [19, 6] and smooth digraphs [12, 2]). Various methods, combinatorial (graph-theoretic), logical and universal-algebraic have been brought to bear on this classification project, with many remarkable consequences. A conjectured delineation for the dichotomy was given in the algebraic language in [7].

Complexity classifications for QCSPs appear to be harder than for CSPs. Indeed, a classification for QCSPs will give a fortiori a classification for CSPs (if $B \uplus K_1$ is the disjoint union of B with an isolated element, then QCSP($B \uplus K_1$) and CSP(B) are polynomially equivalent). Just as CSP(B) is always in NP, so QCSP(B) is always in Pspace. However, no overarching polychotomy has been conjectured for the complexities of QCSP(B), as B ranges over finite structures, but the only known complexities are P, NP-complete and Pspace-complete. It seems plausible that these complexities are the only ones that can be so obtained (for more on this see [9]).

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In this paper we study the complexity of QCSP(H), where H is a semicomplete digraph, i.e. an irreflexive graph so that for each distinct vertices x_i and x_j at least one of $x_i x_j$ or $x_j x_i$ (and possibly both) is in $E(H)$. We prove that each such problem is either in P, is NP-complete or is Pspace-complete. In some respects, our paper is a companion to the classifications for partially reflexive forests [16] and partially reflexive cycles [14], however our work here differs in two important ways. Firstly, this classification is a complete trichotomy instead of a partial classification between P and NP-hard. Secondly, this classification uses the algebraic method to derive hardness results, whereas in [16, 14] surjective polymorphisms appear only for tractability. Indeed, we believe our use of the algebraic method here is the most complex so far for any QCSP trichotomy complexity classification. The first published QCSP trichotomy appeared in (the preprints of) [5] and used relatively straightforward application of the algebraic method pioneered in the same paper. Subsequently, a combinatorial QCSP trichotomy appeared, essentially for irreflexive pseudoforests, in [17]. The task to unite [17, 16, 14], with the spirit of [10], to a QCSP trichotomy for partially reflexive pseudoforests, remains open-ended and ambitious. Two other notable trichotomies have appeared in the QCSP literature in the form of [3] and [4], though both are slightly unorthodox. The former deals with a variant of the QCSP, which allows for relativisation of the universal quantifier, and the latter deals with infinite equality languages.

Our work follows in the spirit of the CSP dichotomy for semicomplete digraphs given long ago in [1]. What we uncover is that the semicompletes with at most one cycle, whose CSPs are in P as per [1], beget QCSPs which remain in P. However, of the semicompletes with more than one cycle, whose CSPs are NP-complete, some produce QCSPs of maximal complexity while others remain no more than NP-complete. Our classification is as follows.

Theorem 1. *Let H be a semicomplete digraph.*

- *If H contains at most one cycle then QCSP(H) is in P, else*
- *H contains a source and a sink and QCSP(H) is NP-complete, else*
- *QCSP(H) is Pspace-complete.*

The tractability results, membership for both P and NP, are relatively straightforward and date back to the last author's 2006 Ph.D. [15]. The natural conjecture was made (not in print) for the trichotomy but repeated efforts to settle it combinatorially failed. The present work arose from a discussion in Dagstuhl about two conjectures involving an algebraic approach, which had always been deemed appropriate as semicomplete digraphs are cores for which all polymorphisms are surjective. The first of these conjectures sought to deal with a large subclass of the semicompletes conjectured to be Pspace-complete, those with neither source nor sink (termed smooth). If it could be proved that all polymorphisms of smooth semicompletes with multiple (i.e. more than one) cycles are essentially unary, then it would be known from [5] that the corresponding QCSP is Pspace-complete. The largest part of this paper is in proving this result. The remaining cases are where there is more than only one cycle and no source (dually resp., sink) but there is a sink (dually resp., source). Suppose then, w.l.o.g, that H^{+m} is built from a smooth semicomplete with multiple cycles H by iteratively adding m sinks. Suppose K_n is the irreflexive n-clique and let K_n^{+m} be the same graph with m sinks iteratively added. The second Dagstuhl conjecture held that, just as the polymorphisms $\mathrm{Pol}(H)$ should be contained in $\mathrm{Pol}(K_n)$, i.e. be only essentially unary, perhaps $\mathrm{Pol}(H^{+m})$ should be contained in $\mathrm{Pol}(K_n^{+m})$, and that would be enough to prove Pspace-completeness for the corresponding QCSP. This conjecture turned out to be false, but some substitute digraphs for K_n in this position were found and so the complexity result follows nonetheless.

As previously stated, the bulk of our work is in proving all smooth semicomplete digraphs with multiple cycles have only essentially unary polymorphisms. It is easy to see this is not true of any of the other semicompletes, for each of which a simple ternary essential polymorphism (i.e. one that is not essentially unary) may be given. Thus, we in fact give another, algebraic, classification.

Theorem 2. *Let H be a semicomplete digraph. If H is smooth and not itself a cycle, then H admits only essentially unary polymorphisms; otherwise H has an essential polymorphism.*

This may be seen as the first part of a larger research program, beginning with semicomplete digraphs, which may continue eventually to larger classes. For example, it is known precisely which smooth digraphs have a weak near unanimity polymorphism [2] and which digraphs enjoy Mal'cev [8]

This paper is organised as follows. After the preliminaries we deal with upper bounds and essential polymorphisms in Section 3. We then deal with the central topic of those semicompletes which have only essentially unary polymorphisms in Section 4. Finally, we deal with the remaining cases of source-without-sink and sink-without-source in Section 5. For reasons of space most proofs are omitted.

2 Preliminaries

Let $[n] := \{1, \dots, n\}$. All graphs in what follows are directed, that is just a binary relation on a set. We denote *digraphs* by G, H, etc. and their vertex and edge sets by $V(.)$ and $E(.)$ (or $\to$, $\Rightarrow$; where $\leftrightarrow$, $\Leftrightarrow$ indicates double edge), respectively, where we might omit the (.) if this is clear. We switch rather freely between postfix notations, such as $xy \in E$, and infix notations such as $x \to y$. If $v \in H$, then $v^+ := \{x \in V(H) : vx \in E(H)\}$ and $v^- := \{x \in V(H) : xv \in E(H)\}$.

A digraph H is *semicomplete* if it is irreflexive (loopless) and for any two vertices i and j, at least one of ij and ji is an edge of H. If H never has both ij and ji, then it is furthermore a *tournament*. For technical reasons we deny the trivial tournament with a single vertex and no edges. The equivalence relation of strong connectedness is defined in the usual way and its equivalence classes will be called strong components. If the strong component has one element, it is trivial, otherwise nontrivial. We start by noting that, just like in the case of tournaments, in semicomplete graphs the strong components can be linearly ordered, so that there is an edge out of every vertex in a smaller strong component into every vertex of a larger strong component (but never an edge going the other way, obviously).

The problems CSP(H) and QCSP(H) each take as input a sentence Φ, and ask whether this sentence is true on H. For the former, the sentence involves the existential quantification of a conjunction of atoms – *primitive positive* (pp) logic. For the latter, the sentence involves the arbitrary quantification of a conjunction of atoms – *positive Horn* (pH) logic. It is well-known, for finite H, that CSP(H) and QCSP(H) are in NP and Pspace, respectively.

The *direct product* $G \times H$ of two digraphs G and H has vertex set $\{(x, y) : x \in V(G), y \in V(H)\}$ and edge set $\{(x, u)(y, v) : x, y \in V(G), u, v \in V(H), xy \in E(G), uv \in E(H)\}$. Direct products are (up to isomorphism) associative and commutative. The kth power G^k of a graph G is $G \times \ldots \times G$ (k times). A homomorphism from a graph G to a graph H is a function $h : G \to H$ such that, if $xy \in E(G)$, then $h(x)h(y) \in E(H)$. A *k-ary polymorphism* of a graph H is a homomorphism from H^k to H. A polymorphism f is *idempotent* when, for all x, $f(x, \ldots, x) = x$. An operation $f : H^k$ to H is termed *essentially unary* if there is a unary operation g and co-ordinate i so that $f(x_1, \ldots, x_k) = g(x_i)$. If f is not essentially unary then we describe f as *essential*.

A digraph is a *core* if all of its endomorphisms are automorphisms. All finite semicomplete digraphs are cores, for which all polymorphisms are surjective. For cores it is well-known the constants are pp-definable up to automorphism. That is, if H^c is H with all constants named, and H is a core, then CSP(H) and CSP(H^c) are poly time equivalent; and the same applies to the QCSP. A similar argument may be given in the algebraic language and the implication is that we may as well assume all the polymorphisms of a semicomplete digraph H are idempotent (because this is true for H^c which is actually the structure we will be working on).

The now-celebrated *algebraic approach* to CSP rests on one half of a Galois correspondence, where it is observed that the relations that are invariant under (preserved by) the polymorphisms of H are precisely the relations that are pp-definable in H. For QCSP, we obtain a similar characterisation substituting surjective polymorphisms for polymorphisms and pH for pp. The consequence of this is that if the polymorphisms (resp., surjective polymorphisms) of H are contained as a subset of those of H', then there is a poly time reduction from CSP(H') to CSP(H) (resp., QCSP(H') to QCSP(H)); that is, the polymorphisms control the complexity.

If Φ is an input for QCSP(H) with quantifier-free part φ, then with this we associate the digraph D_φ whose vertices are variables of φ and edges are given by the atoms in φ. If Φ is existential, i.e. also an input to CSP(H), then the relationship between Φ and D_Φ is that of canonical query to canonical database [13].

In a digraph, a *source* (resp., *sink*) is a vertex with in-degree (resp. out-degree) 0. A digraph with no sources or sinks is called *smooth*. In a semicomplete graph, a source s (resp., sink t) satisfies, for all $x \neq s$ (resp., $x \neq t$), $xs \notin E(H)$ and $sx \in E(H)$ (resp., $tx \notin E(H)$ and $xt \in E(H)$). A digraph may have multiple sources or sinks, but a semicomplete may have at most one of each. If H is a digraph, then let H^{+j} be H with, iteratively, j sinks added (i.e. each time we add a sink we make it forward-adjacent to each existing vertex). Let us label these added sinks, in order, $t_1, \ldots, t_j$ (thus t_j is the unique sink of H^{+j}). Similarly, let $\mathcal{H}^{-j}$ be $\mathcal{H}$ with j sources added. When the j is omitted it is presumed to be 1.

We mention some special semicomplete graphs that will appear in the paper. K_n is the irreflexive complete graph (clique) on vertex set $[n]$. For $i \neq j \in [n]$, K_n has both edges ij and ji. DC_3 is the directed 3-cycle. Let T_n be the transitive tournament on $[n]$ with the natural order $<$ corresponding to the edge relation (i.e. $ij \in E(T_n)$ iff $i < j$).

3 Complexity upper bounds and Essential polymorphisms

The main results of this section date back to the third author's Ph.D. [15] (available from his website) and are presented there combinatorially and in much fuller detail. The first is very straightforward.

Proposition 1. *Let H be a digraph with both a source s and a sink t, then QCSP(H) is in NP.*

Proof. Let Φ be an input to QCSP(H) with quantifier-free part φ. Suppose φ has an atom v_iv_j so that Φ quantifies v_i universally, then Φ is a no-instance since φ will never be satisfied when v_i is evaluated as t. Dually, we may assume φ has no atom v_iv_j so that Φ quantifies v_j universally; and we find that Φ can not contain universally quantified variables involved in atoms of φ. Thus, we may ignore universally quantified variables and evaluate Φ as an input to CSP(H) in NP.

We now turn our attention to the poly time cases.

Proposition 2. *For all $n \geq 1$, QCSP(T_n) is in P.*

Proof. The ternary median function $f(x, y, z) = \mathrm{med}(x, y, z)$ is a polymorphism of T_n which is a majority operation. The tractability of QCSP(T_n) follows from [5].

It is well-known that QCSP(K_2) and QCSP(DC_3) admit a majority polymorphism and are therefore in P (see [5]). We are now interested in the semicomplete graphs K_2^{+j}, K_2^{-j}, DC_3^{+j} and DC_3^{-j} (for $j > 0$). Proof of the following appears in the appendix.

Proposition 3. *For $j \geq 0$, each of QCSP(K_2^{+j}), QCSP(K_2^{-j}), QCSP(DC_3^{+j}) and QCSP(DC_3^{-j}) are in P.*

We now deal with the semicompletes that admit essential polymorphisms.

Proposition 4. *If H is a semicomplete digraph with at most one cycle or a source or a sink, then H admits an essential polymorphism.*

Proof. It was noted in the proof of Proposition 2 that the transitive tournaments admit a median polymorphism. Afterwards it was noted further that K_2 and DC_3 admit majority polymorphisms (and indeed the median may be used here).

Let H be a semicomplete digraph and recall H^+ to be the same digraph with a sink t added, to which all other vertices have a forward edge. Then H has the polymorphism $f(x, y, z) = x$, unless ($y = t$ or $z = t$) in which case $f(x, y, z) = t$. It follows that semicompletes with sink admit an essential polymorphism. The result for semicompletes with a source is symmetric and the result follows.

4 Semicompletes with essentially unary polymorphisms

Theorem 3. *Let H be a smooth semicomplete digraph with precisely two strong components. Then all idempotent polymorphisms of H are projections.*

Theorem 4. *Let H be a smooth semicomplete digraph with two non-trivial strong components. Then all idempotent polymorphisms of H are projections.*

Theorem 5. *Let H be a smooth semicomplete digraph with more than two strong components. Then all idempotent polymorphisms of H are projections.*

We sum these up in the following corollary.

Corollary 1. *Let H be a smooth semicomplete digraph that is not strongly connected. Then all idempotent polymorphisms of H are projections.*

4.1 The strongly connected case

Definition 1. *A subset $L \subset V$ is nice if the induced subgraph on L is strongly connected and all idempotent polymorphisms of G restrict to L as projections.*

Lemma 1. *Let L be a nice subset of V and let v be a vertex such that $v^+ \cap L \neq \emptyset \neq v^- \cap L$. Then $L \cup \{v\}$ is nice.*

Lemma 2. *Let $L = \{a, b\}$ be compatible with (i. e. closed under) the idempotent polymorphisms of G and let $a \to b \to a$. If $v \in V \setminus L$ is such that $v^+ \cap L \neq \emptyset \neq v^- \cap L$ and f is an n-ary idempotent polymorphism of G, then there exists i, $1 \leq i \leq n$, such that on the subset $\{a, b, v\}$ the restriction of f is equal to the ith projection.*

A *congruence* of a tournament $(V, \to)$ is an equivalence relation ρ on V such that for all (x_1, x_2), $(y_1, y_2) \in \rho$ such that $(x_1, y_1) \notin \rho$, $x_1 \to y_1$ iff $x_2 \to y_2$. If ρ is a congruence of the tournament $T = (V, \to)$, then the *factor tournament* T/ρ is the tournament $(V/\rho, \Rightarrow)$, where $a/\rho \Rightarrow b/\rho$ iff $a/\rho \neq b/\rho$ and $a \to b$.

We also introduce the interval notation for a digraph $G = (\{a_1, a_2, \ldots, a_n\}, \to)$ with the fixed Hamiltonian cycle $a_1 \to a_2 \to \ldots \to a_n \to a_1$: $[a_i, a_j]$ is the set of all vertices that are traversed by shortest path starting at a_i, ending at a_j and which uses only the directed edges of the Hamiltonian cycle. For instance, $[a_2, a_1] = \{a_1, a_2, \ldots, a_n\}$, while $[a_1, a_2] = \{a_1, a_2\}$. We also define $[a_i, a_j) := [a_i, a_j] \setminus \{a_j\}$, $(a_i, a_j] := [a_i, a_j] \setminus \{a_i\}$ and $(a_i, a_j) := [a_i, a_j] \setminus \{a_i, a_j\}$.

Definition 2. *Let $T = (\{a_1, \ldots, a_n\}, \to)$ be a strongly connected tournament with the fixed Hamiltonian cycle $C = a_1 \to a_2 \to \ldots \to a_n \to a_1$, where $n \geq 3$. T is* locally transitive *with respect to the cycle C iff there exists a function $\varphi_T : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that:*

1. *$\varphi_T(i) \notin \{i-1, i\}$ and $\varphi_T(1) \notin \{1, n\}$,*
2. *$a_i^+ = (a_i, a_{\varphi_T(i)}]$ and*
3. *$a_{\varphi_T(i+1)} \in [a_{\varphi_T(i)}, a_i)$ and $a_{\varphi_T(1)} \in [a_{\varphi_T(n)}, a_n)$.*

In particular, since the locally transitive tournament T is semicomplete, we get that $a_{\varphi_T(i)+1} \to a_i$ and from the definition above follows that

$$(4) \qquad a_i \to a_{i+1}, \qquad (a_{i+1} \to a_{\varphi_T(i)} \text{ or } a_{i+1} = a_{\varphi_T(i)}) \quad \text{and} \quad a_i^+ \backslash \{a_{i+1}\} \subseteq a_{i+1}^+$$

(where the addition here is modulo n, so $n+1 = 1$). Note also that local transitivity depends on the fixed Hamiltonian cycle C. It is easy to construct five-element Hamiltonian tournament which is locally transitive with respect to one of its Hamiltonian cycles, but not with respect to another.

We will use the easier notation for a locally transitive tournament T when the vertex set is $\{1, 2, \ldots, n\}$, where we will understand, unless otherwise stated, that the fixed Hamiltonian cycle is $1 \to 2 \to \ldots \to n \to 1$, and $a_i = i$, so we will have $(\varphi_T(i) + 1) \to i$ instead of $a_{\varphi_T(i)+1} \to a_i$ et cetera.

Definition 3. *A locally transitive tournament $T = (\{1, \ldots, n\}, \to)$ is* regular *iff $n = 2k+1$ for some positive integer k and for all $1 \leq i < j \leq 2k+1$, $i \to j$ iff $j - i \leq k+1$ (otherwise $j \to i$). In other words, in the unique (up to isomorphism) regular locally transitive tournament with $2k+1$ vertices, $\varphi_T(i) = i+k$ if $i \leq k+1$, and $\varphi_T(i) = i-k-1$ if $i > k+1$.*

Lemma 3. *Let $T = (\{1, \ldots, n\}, \to)$ be a locally transitive tournament such that φ_T is a permutation of $\{1, \ldots, n\}$. Then T is regular.*

Definition 4. *The semicomplete graph $G_T = (V, E)$ will be called a P-graph parametrized by the locally transitive tournament $T = (\{1, \ldots, n\}, \to)$ if there exists a partition ρ of the vertex set V into nonempty subsets $A_1, \ldots, A_n$ such that for all $i \neq j$ and all $a \in A_i$ and $b \in A_j$, $ab \in E$ iff $i \to j$ in T.*

Theorem 6. *Every idempotent polymorphism f of a P-graph G_T parametrized by the locally transitive tournament T is a projection, except when G_T is the 3-cycle.*

Lemma 4. *Let $G = (V, \to)$ be a strongly connected semicomplete graph which contains at least one 2-cycle. Then for each 2-cycle $a \leftrightarrow b$ in G, the set $\{a, b\}$ is closed with respect to all idempotent polymorphisms of G and each binary idempotent polymorphism of G restricted to $\{a, b\}$ is a projection.*

Definition 5. *Let $G = (V, \to)$ be a strongly connected semicomplete graph. We say that L* splits *G if $\emptyset \neq L \subsetneq V$ is a subset with the following properties:*

1. $\{L, L^+, L^-\}$ is a partition of V and
2. for any 2-cycle $a \leftrightarrow b$ in G, $\{a, b\}$ is contained in one of L, L^- or L^+.

Lemma 5. *Let $G = (V, \rightarrow)$ be a strongly connected semicomplete graph which is not a cycle. Let L_0 be either a 2-cycle or a nice subset of V. Then either all idempotent polymorphisms of G are projections, or there exists a subset $L \subset V$ such that L splits G, $L_0 \subset L$ and either the induced subgraph on L is a 2-cycle, or L is nice.*

Lemma 6. *Let $G = (V, \rightarrow)$ be a strongly connected semicomplete graph which is not a P-graph and let L split G. Then there exist vertices $a_0, a_1, b_0 \in V$ such that $a_1 \leftarrow a_0 \rightarrow b_0 \rightarrow a_1$ and that either*

1. *$b_0 \in L^-$ and a_0, a_1 are in the same strong component, or two consecutive strong components, of the induced subgraph on L^+, or*
2. *$b_0 \in L^+$ and a_0, a_1 are in the same strong component, or two consecutive strong components, of the induced subgraph on L^-.*

Lemma 7. *If a strongly connected tournament $G = (V, \rightarrow)$ is not a P-graph and for all $v \in V$, all strong components of the induced subgraphs on v^+ and on v^- are of sizes 1 or 3, then there is a 3-cycle $a \rightarrow b \rightarrow c \rightarrow a$ in G such that all idempotent polymorphisms of G restrict to $\{a, b, c\}$ as projections.*

Theorem 7. *A strongly connected semicomplete digraph which is not a cycle has all its idempotent polymorphisms being projections.*

Proof. We prove it by an induction on $|V| = n$. By Theorem 6, if G is a P-graph, we are done, so we assume that G is not a P-graph. For $n = 2$ the only semicomplete digraph must be a cycle. If $n = 3$ and G is not a cycle, then there is a 2-cycle $a \leftrightarrow b$ in G, and the third vertex c must satisfy either $a \rightarrow c \rightarrow b$ or $b \rightarrow c \rightarrow a$ (possibly even both!), so by Lemma 4 and Lemma 2 all idempotent polymorphisms are projections. Also, if $n = 4$, then G is a P-graph parametrized by the 3-cycle if G is the only 4-element strongly connected tournament or in the case when $V = \{a, b, c, d\}$ has exactly one 2-cycle $a \leftrightarrow b$, $c \in \{a, b\}^+$ and $d \in \{a, b\}^-$. Otherwise, from Lemmas 4, 2 and 1 follows that all idempotent polymorphisms of G are projections.

Now assume that $n > 4$ and that the Theorem holds in all strongly connected semicomplete graphs with fewer than n vertices. If there exists a 2-cycle $a \leftrightarrow b$, then we set $L_0 = \{a, b\}$. Otherwise, G is a tournament, and if there exists any vertex $v \in V$ and a strong component L_0 of the induced subgraph on v^- or on v^+ such that $|L_0| > 3$, then L_0 is clearly pp-definable with constants in G, so L_0 must be nice by the inductive assumption. Finally, if G is a tournament and for all $v \in V$ all strong components of the induced subgraphs on v^- and on v^+ have at most three elements, then by Lemma 7 follows that there is a three element subset L_0 which is nice.

Let L be a maximal nice subset of V such that $L_0 \subset L$. If $L \neq V$, then by Lemma 5, L splits G. Now from Lemma 6 follows that either a strong component L' of the induced subgraph on a_0^+ contains $L \cup \{a_1, b_0\}$ (if (1) of Lemma 6 holds), or that a strong component L' of the induced subgraph on a_1^- contains $L \cup \{a_0, b_0\}$ (if (2) of Lemma 6 holds). Either way, L' is pp-definable with constants in G, $L \subsetneq L' \subsetneq V$ and the induced subgraph on L' is strongly connected, so by the inductive assumption L' is nice. This contradicts the assumed maximality of L. So, the only alternative is $L = V$, but then the Theorem holds by niceness of L.

Our main complexity result now follows from [5].

Corollary 2. *If H is a smooth semicomplete digraph with more than one cycle, then QCSP(H) is Pspace-complete.*

5 Remaining semicomplete digraphs

Recall $\mathcal{T}_n$ to be the transitive tournament on $[n]$ with the natural order $<$ corresponding to the edge relation. Let $\overline{\mathcal{T}_n}$ be $\mathcal{T}_n$ with the extant edge $E(1, n)$ augmented by $E(n, 1)$, i.e. this becomes a double-edge. Let $\mathcal{K}_{2\to 2}$ be the semicomplete graph built from disjoint copies $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{K}_2$ with all edges added from $\mathcal{H}_1$ to $\mathcal{H}_2$. More generally, let $\mathcal{K}_{2\to 1^k \to 2}$ be the semicomplete graph built from disjoint copies $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{K}_2$ with a transitive tournament T_k inbetween.

5.1 Some Pspace-hardness results

Proposition 5. *For each $k > 0$, QCSP($K_{2\to 2}$) and QCSP($K^+_{2\to 2}$) are Pspace-complete.*

Corollary 3. *Let $G = (V, \to)$ be a finite digraph without loops. Let G contain either*

1. *a copy of $K_{2\to 2}$ such that $a \leftrightarrow b \to c \leftrightarrow d$ such that any automorphism of this copy extends by the identity map to an automorphism of G and moreover, $a^+ \cup b^+ = V$, or*
2. *a copy of K_3, $a \leftrightarrow b \leftrightarrow c \leftrightarrow a$ such that any permutation of $\{a, b, c\}$ extends by the identity map to an automorphism of G and moreover $a^+ \cup b^+ = a^+ \cup c^+ = b^+ \cup c^+ = V$,*

then QCSP(G) is Pspace-complete.

Proposition 6. *For $n \geq 3$, both QCSP($\overline{T}_n$) and QCSP($\overline{T}^+_n$) are Pspace-complete.*

Proposition 7. *For any digraph H, QCSP(H^+) and QCSP$_{[\exists/H]}$(H^+) are equivalent.*

For H a subset of the domain of the structure H', let QCSP$_{[\exists/H]}(H')$ be the variant of QCSP(H') in which the existential variables are restricted to being chosen from H.

Proposition 8. *Let H be a digraph. For each $j > 1$ there exists a polytime reduction from QCSP$_{[\exists/H]}$(H^+) to QCSP(H^{+j}).*

Corollary 4. *For any digraph H and each $j > 1$, QCSP(H^+) reduces to QCSP(H^{+j}).*

Corollary 5. *For each $j > 0$, QCSP($\overline{T}^{+j}_n$) and QCSP($K^{+j}_{2\to 2}$) are both Pspace-complete.*

5.2 The algebraic part

Definition 6. *Let $G = (V, \to)$ be a directed graph. We define the relation $\preceq_G$ on V by $x \preceq_G y$ iff $x^- \subseteq y^-$.*

Proposition 9. *Assume that G is semicomplete. Then $\preceq_G$ is a partial order, $\preceq_G$ has the largest element t iff t is a sink, and dually for least elements and sources.*

Lemma 8. *Let $G = (V, \to)$ be a semicomplete graph without sources, but with the sink t. Let $f : V^m \to V$ be any idempotent mapping such that its restriction to $V \setminus \{t\}$ is the first projection. f is a polymorphism of G iff for all $b_1, b_2, \ldots, b_m \in V$, $b_1 \preceq_G f(b_1, b_2, \ldots, b_m)$.*

Definition 7. *Let $G = (V, E)$ be a digraph. We define the partition of the vertex set V into V^G_{min}, V^G_{max}, V^G_{both} and V^G_{none} so that all vertices in V^G_{max} are minimal, but not maximal, in the order $\preceq_G$, all vertices in V^G_{min} are maximal, but not minimal, in the order $\preceq_G$, all vertices in V^G_{both} are both minimal and maximal in the order $\preceq_G$, while vertices in V^G_{none} are neither minimal nor maximal in the order $\preceq_G$. When the digraph G is understood, we will omit the superscript ^{G}.*

Definition 8. *Let $G = (V, E)$ be a digraph. We define the digraph $S(G) = (V, \to)$ by:*

1. *For all $x, y \in V_{max} \cup V_{both}$, $x \leftrightarrow y$,*
2. *For all $x, y \in V_{min}$, $x \leftrightarrow y$,*
3. *For all $x, y \in V_{none}$, $x \to y$ iff $E(x, y)$.*
4. *For all $x \in V_{min}$ and $y \in V_{none} \cup V_{max}$, $x \to y$, but $\neg y \to x$,*
5. *For all $x \in V_{none}$ and $y \in V_{max}$, $x \to y$, but $\neg y \to x$,*
6. *For all $x \in V_{both}$ and $y \in V_{none} \cup V_{min}$, $x \to y$, but $\neg y \to x$.*

Proposition 10. *$V^{S(G)}_{min} = V^G_{min}$, $V^{S(G)}_{max} = V^G_{max}$, $V^{S(G)}_{both} = V^G_{both}$ and $V^{S(G)}_{none} = V^G_{none}$. Consequently, $S(S(G)) = S(G)$.*

Proposition 11. *A permutation α of the vertex set V of the digraph $G = (V, \to)$ (more generally, universe A of a finite relational structure) is an automorphism iff it is structure-preserving.*

Lemma 9. *The following statements hold for any digraph G:*

1. *$Aut(G) \subseteq Aut(V, \preceq_G)$,*
2. *$Aut(G) \subseteq Aut(S(G))$,*
3. *$\preceq_G \subseteq \preceq_{S(G)}$ and*
4. *If G is smooth and semicomplete, then so is $S(G)$.*
5. *If G is not a cycle and semicomplete, then neither is $S(G)$.*

Corollary 6. *Let $G = (V, E)$ be a smooth semicomplete digraph which is not a cycle. Then $Pol(G^+) \subseteq Pol(S(G)^+)$.*

Definition 9. *Let $G = (V, E)$ be a digraph. We define the digraph $L(G)$ on the set V in the following way:*

1. *For all $x \in V_{both} \cup V_{min}$ and $y \in V_{none} \cup V_{max}$, $x \to y$, but $\neg y \to x$,*
2. *For all $x \in V_{none}$ and $y \in V_{max}$, $x \to y$, but $\neg y \to x$,*
3. *For all $x, y \in V_{min} \cup V_{both}$, $x \leftrightarrow y$,*
4. *For all $x, y \in V_{none}$, $x \to y$ iff $E(x, y)$,*
5. *For all $x, y \in V_{max}$, $x \leftrightarrow y$.*

The next Lemma follows directly from Definition 9.

Lemma 10. *Let G be a digraph. Either $V = V^G_{both} = V^{L(G)}_{both}$, or $V^{L(G)}_{min} = V^G_{both} \cup V^G_{min}$, $V^{L(G)}_{none} = V^G_{none}$, $V^{L(G)}_{max} = V^G_{max}$ and $V^{L(G)}_{both} = \emptyset$.*

Corollary 7. *Let $G = (V, E)$ be a smooth semicomplete digraph which is not a cycle. Then $Pol(S(G)^+) \subseteq Pol(L(G)^+)$.*

Theorem 8. *Let $G = (V, E)$ be a smooth semicomplete digraph which is not a cycle. Then QCSP(G^{+j}) is Pspace complete for all $j > 0$.*

Corollary 8. *If H is semicomplete with more than one cycle and either: 1.) a sink but no source, or 2.) a source but no sink, then QCSP(H) is Pspace-complete.*

Proof. Case 1 is taken care of by Theorem 8 and Case 2 is symmetric.

6 Conclusion

We can now piece together proofs of our central theorems.

Proof (of Theorem 1). The cases in P follow from Propositions 2 and 3. The NP upper bound follows from Proposition 1 and the NP lower bound follows from [1]. All (finite-domain) QCSPs are in Pspace so, finally, the Pspace-hard cases follow from Corollaries 2 and 8.

Proof (of Theorem 2). From Proposition 4 and Theorem 7.

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