On self–normalised sums. Supplement.

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Math. Methods Statist., 2002, v. 11, No 2, 256-258.

Abstract

This supplement aims to correct a misprint and clarify the point about the accuracy of Berry–Esseen–type inequalities for self–normalised sums and Student's statistic established in [1].

AMS 1991 Subject Classification: primary 62E20, secondary 62G05, 60F05.
Key words and phrases: self–normalised sums, Student's statistic, accuracy of normal approximation.

Let $X, X_1, X_2, ...$ be a sequence of independent and identically distributed random variables. Denote $t_n = \sum_{i=1}^n X_i \Big/ \sqrt{\sum_{i=1}^n X_i^2}$ (here and in the sequel we use the notation from [1]).

Theorems 9 and 10 in [1] establish uniform and non–uniform Berry–Esseen–type inequalities for self–normalised sums t_n and Student's statistic.

The lower and upper bounds for $\mathbb{P}\,(t_n < x) - \Phi(x)$ in [1] are given with explicit constants, and the only moment assumption is the finiteness of the 3rd moment. The upper bound is of order $n^{-1/2}$. The lower bound is of order $n^{-2/7}$; it is of order $n^{-1/2}$ if the 4th moment is finite. In particular,

$$\sup_x |\mathbb{P}(t_n < x) - \Phi(x)| \le C_* \frac{\mathbb{E}|X|^3}{\sqrt{n}} + \frac{8|\mathbb{E}X|^3}{e\sqrt{2\pi n}} + \frac{\sqrt{\mathbb{D}X^2}}{\sqrt{n}} \left(\frac{2}{\pi e}\right)^{1/4} + o(n^{-1/2}), \quad (1)$$

where C_* be the constant from the uniform Berry–Esseen inequality (the term $8\mathbb{E}|X|^3/(e\sqrt{2\pi n})$ in [1], p. 424, may be replaced by $8|\mathbb{E}X|^3/(e\sqrt{2\pi n})$).

The author has received a feedback that the bounds of Theorems 9 and 10 are not easy to understand. One distinguished scientist wrote that (1) was the only inequality with explicit constants he was able to find in [1].

The aim of this letter is to clarify that point and correct a misprint. We show that Theorems 9 and 10 provide bounds with explicit constants. Moreover, the bounds are sharp in the following sense: for a class of probability distributions, the estimates are asymptotically as sharp as those of the uniform and non–uniform Berry–Esseen inequalities.

There is a misprint in the definition of $R(t,a,b)$ on page 423: "c" in the denominator must be erased. Examples below illustrate the sharpness of the Berry–Esseen–type inequalities for self–normalised sums.

Example 1. Let $\mathbb{P}(X=-1)=\mathbb{P}(X=1)=1/2$. Then Theorem 9 with $\varepsilon=0$ and $N=1$ yields

$$\Delta_n \equiv \sup_x |\mathbb{P}(t_n<x)-\Phi(x)| \le C_* \big/ \sqrt{n}\,. \tag{2}$$

The right–hand side of (2) coincides with that of the Berry–Esseen theorem.

Put $\varepsilon_{n,x}=0$, and notice that

$$\rho_n=p_n=\delta_{n,x}=\gamma_n^+=0\,,\; R_n^+(x,3)=C_+n^{-1/2}\,,$$

where C_+ the constant from the non–uniform Berry–Esseen inequality. Theorem 10 yields

$$|\mathbb{P}(t_n<x)-\Phi(x)| \le C_+n^{-1/2}(1+x^3)^{-1}\,. \tag{3}$$

The right–hand side of estimate (3) coincides with that of the non–uniform Berry–Esseen inequality.

Example 2. Let $\mathbb{P}(X=-N)=\mathbb{P}(X=N)=1/(2N^2)$, $\mathbb{P}(X=0)=1-N^{-2}$. Denote $R_{n,N}=\frac{N}{\sqrt{n}}\left[C_*^{1/3}+C_+^{1/3}N/\sqrt{n}\right]^3$ and $r_{n,N}=\frac{24\sqrt{6}\,N^2}{e\sqrt{\pi e}\,n}$. It is easy to check that

$$\mathbb{E}X=0,\ \mathbb{E}|X|^m=N^{m-2}\ (m\in\mathbb{N}),\ \rho_n=m_0=0\,,$$

$$\beta_n=e^{-n/(36N^2)}\,,\ \max\{R_n(3,1,1);R_n(3,2,2)\}\le R_{n,N}\,,$$
$$r_n<2r_{n,N},\ \mathbb{P}\Big(\sum\nolimits_{i=1}^n (X_i^2-1)/n>\sqrt{2\varepsilon}\Big)\le e^{-\varepsilon n/(2N^2)}\,.$$

Theorem 9 gives

$$\Phi(x(1-\varepsilon))-R_{n,N}-2r_{n,N}-e^{-\varepsilon n/(2N^2)} \le \mathbb{P}(t_n<x)\le \Phi(x)+R_{n,N}+r_{n,N}\,. \tag{4}$$

Assume that $N=N(n)\to\infty$, $N^2/n\to 0$. With $\varepsilon=2n^{-1}N^2\ln(n/N^2)$, (4) implies

$$\Delta_n \le C_*Nn^{-1/2}+O\Big(n^{-1}N^2\ln\big(n/N^2\big)\Big)\,.$$

In other words, estimate (4) is asymptotically as sharp as that of the Berry–Esseen theorem.

In order to apply Theorem 10, note that $\gamma_n^+\le\gamma_{n,x}^+$ and $R_n^+(x,3)\le R_{n,x}^+$, where

$$\gamma_{n,x}^+=9x^2N^2/25n\,,\ R_{n,x}^+=C_+Nn^{-1/2}\left(1+3xN/(5\sqrt{n})\right)^3.$$

Besides, $\mu_3^{>} = 0$ if $x \geq 6N/\sqrt{n}$. Theorem 10 yields

$$\Phi\Big(x(1-\varepsilon)(1-\gamma^+_{n,x})\Big) - R^+_{n,x}\left(1 + x^3(1-\gamma^+_{n,x})^3/(1-\varepsilon)^3\right)^{-1} - e^{-\varepsilon n/(2N^2)}$$

$$\leq \mathbb{P}(t_n < x) \leq \Phi(x) + R^+_{n,x}\left(1 + x^3(1-\gamma^+_{n,x})^3\right)^{-1} + e^{-n/(36N^2)} \qquad (5)$$

as $6N/\sqrt{n} \leq x \leq \sqrt{n}/(3N)$. With $\varepsilon = 2n^{-1}N^2 \ln(nN^{-2})(2+x^2)$, inequality (5) implies

$$-C_+\Big(1 + 3xN/(5\sqrt{n})\Big)^3 - v_n n^{-1}N^2 \ln(n/N^2) \leq (1+x^3)[\mathbb{P}(t_n < x) - \Phi(x)]\sqrt{n}/N$$

$$\leq C_+\Big(1 + 3xN/(5\sqrt{n})\Big)^3 (1+x^3)\left(1 + x^3(1-\gamma^+_{n,x})\right)^{-1} + v_n n^{-1}N^2\,,$$

where $v_n \to 0$ uniformly in $x \in [6N/\sqrt{n}; \sqrt{n}/(3N)]$. Let $\{u_n\}$ be a sequence of positive numbers such that $u_n \to 0$ and $u_n\sqrt{n}/N \to \infty$. Then

$$(1+x^3)|\mathbb{P}(t_n < x) - \Phi(x)|\sqrt{n}/N \leq C_+ + o(1) \qquad (6)$$

uniformly in $x \in [0; u_n\sqrt{n}/N]$.

In the expanding interval $[0; u_n\sqrt{n}/N]$, estimate (6) is asymptotically as sharp as that of the non–uniform Berry–Esseen bound. We notice in [2] that a non–uniform Berry–Esseen–type inequality for self–normalised sums may not, in general, hold on the whole line.

References

[1] S.Y. Novak (2000) On self–normalised sums. — Math. Methods Statist., v. 9, No 4, 415–436.

[2] S.Y. Novak (2001) On self–normalised sums and Student's statistic. — Brunel University: Technical Report No TR/18/01.