CASE STUDY

Thematic problem solving: a case study on an approach to teaching problem solving in undergraduate mathematics

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Abstract

Specialist mathematics, statistics and operational research (MSOR) programmes are recognised as intellectually demanding, and require students to formulate, abstract, and solve mathematical problems in a rigorous way. The process of developing the skills to do this well and communicate results can be challenging for learners as it requires a deep understanding of themes in mathematics as well as methods for solving problems. In this article we demonstrate how elements of Freudenthal's Realistic Mathematics Education can be applied to teaching problem solving in undergraduate mathematics programmes. We describe an approach that moves away from standard practices and goes beyond problem solving methods to develop an understanding of common themes in mathematics.

Keywords: Problem solving, mathematical education, realistic mathematical education, cognitive process.

1. Introduction

In this paper we discuss an approach to teaching higher level mathematical problem-solving skills to specialist undergraduate mathematics students. Our approach introduces students to the step-wise problem solving methodology found in Polya (1957), Mason et al. (2010) and to a lesser extent Bransford and Stein (1993). However, our procedure goes beyond these kinds of cognitive training programs and aims to develop students' familiarity and confidence in usage of themes in mathematical proof. We define a theme as an argument or portion of an argument that is common to a number of proofs that students encounter in their studies. This definition stems from the observation that most arguments in mathematics are made up of smaller, common reusable arguments. For example, the standard proof of the uniqueness of the identity element in a group remains essentially unchanged if one replaces a group with almost any algebraic structure. Similarly, in number theory a common theme when studying integers is to rephrase a problem in modulo arithmetic where only finitely many cases need be considered. When dealing with convergent sequences in analysis it is often useful to split the sequence into a finite part and the tail of the sequence.

Each of the themes discussed in the preceding paragraph, and many others, are typically encountered early in undergraduate mathematics degrees. In this article we argue that for students to develop their problem-solving skills they not only need to train their cognitive thinking processes but also need to recognize and collate a library of themes and techniques that are applicable to a wide range of problems. This archive of higher-level themes in mathematics is often overlooked in problem solving literature where the focus is more on models for cognitive problem-solving methods. In fact, it is only in Polya (1957) that we find a substantive discussion of the notion of a theme.

We present in this paper a case-study of a second year specialist mathematics module taught entirely in problem solving workshops where activities are designed, using structures familiar in

Freudenthal’s Realistic Mathematics Education (1968 and 1973) and Moore’s method (Jones, 1977), to develop an understanding and appreciation of themes.

The paper is set out as follows. In sections 2 and 3 we discuss the prevailing discussion surrounding problem solving skills in undergraduate mathematics education. In section 4 we introduce and discuss some of the relevant aspects of Freudenthal’s Realistic Mathematics Education – we argue that themes in mathematics, as introduced above, are as much to do with Freudenthal’s notion of real-life mathematics as the concrete problems he was interested in. Finally in sections 5 and 6 we discuss our approach to developing students’ understanding of themes and our initial findings from teaching these techniques.

2. Problem solving from Polya onwards

Polya’s work on the systematisation of problem-solving methods in mathematics has long been hailed as one of the most important treatises on this topic, (Polya, 1957) – certainly one of the most comprehensive. Its influence on mathematics educators cannot be overstated. In fact its influence reaches beyond mathematics, see for example Bransford and Stein (1993). Polya was one of the first authors to suggest presenting mathematics as an experimental science. He observed:

“Mathematics presented in the Euclidean way appears as a systematic, deductive science; but mathematics in the making appears as an experimental, inductive science.” (Polya, 1957)

Importantly, Polya appreciated that these two sides of mathematics – the rigorous logical argument structure of the definition/theorem/proof inherited from Euclid, and the haphazard experimentation of mathematical discovery – rely one on the other. Implicit in his work is the notion that the student of mathematics cannot hope to progress in one without progressing in the other; indeed, many of Polya’s heuristics rely on experience and advanced knowledge of previous work.

Polya’s model for problem solving specifies four stages: Understanding the problem, devising a plan, carrying out the plan, and looking back. It is the second stage that Polya (1957) largely focussed on and, indeed, is the focus of the current article. A modern practical approach to Polya’s work that deserves a mention is (Mason et al., 2010) that translates Polya’s four stage approach to three “phases of work”: Entry, Attack, and Review. Emphasis is made of the cognitive processes of mathematical problem solving. The authors discuss briefly themes in mathematics, however this is not pursued in detail. Other authors, Bransford and Stein (1993) for example, have argued that Polya’s methodology can be applied to a broader range of problems. However, the authors discuss the importance of specialized subject-specific knowledge and note that *“[o]ur ability to solve problems is not simply equivalent to a set of general problem-solving skills.”* We are particularly interested in this notion of subject specific knowledge. We have found that for mathematics, subject specific knowledge goes far beyond the definitions and proofs encountered, often didactically, in traditional mathematics courses. Indeed, it is our opinion that it is the themes alluded to earlier that are equally important.

Many authors have hypothesised cognitive thinking models for mathematical problem solving. In Mayer (1992), for example, the author proposes a model that specifies five types of knowledge that a student must demonstrate in order to solve a mathematical problem: linguistic knowledge, semantic knowledge (a student’s general knowledge of mathematical facts), schematic knowledge (a student’s knowledge of the topic of the problem and their ability to recognise different types of problem), strategic knowledge (a student’s knowledge of how to use their available knowledge to *“develop a plan”*), and finally procedural knowledge (the student’s knowledge of mathematical manipulation and argument construction). Here Mayer specifically isolates Polya’s second stage. In Kintsch and Greeno (1985) on the other hand the authors’ work on problem solving of arithmetic and algebraic word problems also highlights the importance of schematic knowledge and procedural knowledge. However less emphasis is given to the development of strategies to solve unfamiliar problems. In Reusser (1996) the author avoids the difficulties of solving problems in unfamiliar contexts, proposing a step-wise processing model including the five stages: constructing a propositional representation of the problem, creating a situational model, transforming the situational model into a formal mathematical representation, applying the operations to calculate the solution, and interpreting the solution in a meaningful way.

The current article presents a case study of an approach to problem solving that highlights the character of different subjects in mathematics (for example analysis, algebra etc.) and enables students to develop a library of techniques that provide insight into developing strategies. The proposed strategy utilises a stepwise approach, influenced by Polya's work, as well as Mason et al. but also aims to develop the notion of strategic knowledge (Mayer, 1992). However, our approach differs in the content of workshop sessions from the discursive model, described by Lakatos (1976) for example, and instead uses similar ideas from Realistic Mathematics Education, (Freudenthal, 1968 and 1973), discussed below to discover, study and reflect on themes.

3. Themes in mathematics: have you seen it before?

Proofs that students encounter in undergraduate mathematics, especially during the initial weeks of teaching, rely on what educators often call "tricks". For example, as mentioned in the introduction, to prove a group has unique identity one normally proceeds by assuming that it has two distinct identities, e, and e'. It is then argued that $e = ee'$ since e' is an identity, and therefore that $e = e'$ since e is an identity, thus producing a contradiction. This technique can be mimicked to produce similar proofs of unique identity elements for a number of algebraic structures, for example a vector space, or a field; it is often termed a "trick". The authors would argue that the notion of a trick in mathematics is at best a misrepresentation of mathematical arguments and at worst an untruth. The idea of a trick gives students the impression that it is something they would not be able to think of themselves. We instead recognise this example as a reusable theme in algebra. The different branches of mathematics include numerous examples of themes that students study routinely as part of their mathematics education, such as the examples in the introduction. It is these themes that Mayer (1992) terms strategic knowledge and Polya (1957) is describing when he asks the question: *"Have you seen it before?"* Furthermore, it is knowledge of these themes that professional mathematicians make use of daily to solve problems.

In order to enable students to develop as successful problem solvers it is important to highlight themes in mathematics and have students reflect on, recognise and make use of themes in their problem solving. In this way we depart from theories of problem solving (Kintsche and Greeno, 1985; Reusser, 1996) that prioritise the recognition of problems by their type. Instead we propose studying solutions to problems in order to develop a student's library of themes and tools that can be subsequently applied to solve problems more flexibly. It is from this point of view that we are proposing a technique to develop students' cognition of themes in mathematics and, in particular, their ability to reuse themes from one area in order to better facilitate the solution to problems in others.

4. Realistic mathematics education

In order to develop the library of themes discussed in the previous sections we have designed a number of activities that let students discover tools for themselves. We employ notions of scaffolded learning to enable students to develop their own internal reflective monitor that helps them study proofs and solutions to problems in order to isolate the main arguments employed. In this way activities are designed to teach students to reuse arguments and tools in problem solving.

The main influence in the design of these activities comes from Freudenthal's Realistic Mathematics Education (RME), (Freudenthal, 1968, 1973). Hans Freudenthal developed RME in opposition to the didactic approach to mathematics education that was being exercised throughout Europe and the United States in the 1960s and 1970s, and in particular of the 'new mathematics' of the 1960s. His approach emphasised the development of mathematics curricula, in the way Polya argued, as an *"experimental, inductive science."* For an interesting survey of Freudenthal's work see Gravemeijer and Terwel (2000).

Freudenthal took the point of view that although mathematics as an abstract subject is extremely flexible and hence applicable, it is *"wasted on individuals who are not able to avail themselves of this flexibility,"* (Freudenthal, 1968). However, he argued that simply teaching students what educators felt was *"useful mathematics"* would lead to a narrow knowledge of mathematics and in essence remove the flexibility inherent in mathematics. On the other hand, he argued that teaching pure mathematics and afterwards working through examples of applications was also *"the wrong order"* (Freudenthal, 1968).

Instead Freudenthal developed the notion of mathematizing, of doing mathematics as a human activity; he said,

"[Mathematics as a human activity] is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter." (Freudenthal, 1968).

This approach stood in contrast to other mathematics educators both then and now who often propose a more discursive approach to mathematics education, along the lines of Lakatos (1976) and Polya (1957). RME was therefore developed in order to facilitate mathematizing. In Gravemeijer (1994), (see also Gravemeijer and Terwel, 2000) the author clarifies the characteristics of mathematizing, or *"making more mathematical"*, as techniques:

- for generality: generalizing (looking for analogies, classifying, structuring);
- for certainty: reflecting, justifying, proving (using a systematic approach, elaborating and testing conjectures, etc.);
- for exactness: modelling, symbolizing, defining (limiting interpretations and validity), and;
- for brevity: symbolizing and schematizing (developing standard procedures and notations).

It is notable that these overlap with (Polya, 1957) and also 'specialising' and 'generalising' in (Mason et al., 2010). In Treffers (1987), the author further distinguishes different activities as horizontal and vertical mathematizing. Horizontal mathematizing involves taking a problem and converting it to a mathematical problem, whereas vertical mathematizing involves taking a mathematical problem and reformulating it or understanding it in a deeper way, similar to Polya's understanding the problem, or Mason et al.'s entry phase. In the words of Freudenthal,

"Horizontal mathematizing leads from the world of life to the world of symbols. In the world of life one lives, acts (and suffers); in the other one symbols are shaped, reshaped, and manipulated, mechanically, comprehendingly, reflectingly: this is vertical mathematization. The world of life is what is experienced as reality (in the sense I used the word before), as is a symbol world with regard to abstraction. To be sure the frontiers of these worlds are vaguely marked. The worlds can expand and shrink – also at one another's expense." (Freduenthal, 1991)

5. Activity design

As an example, this case study describes an activity designed to last approximately two hours with the objective of revisiting modulo arithmetic and developing it as a theme in number theory to solve problems concerning the integers.

The session begins with a description of the intended outcomes without reference to modulo arithmetic, namely in terms of tangible knowledge of tools for problem solving. Students, over the two hours, intermittently work through the following activities in order. Having solved one problem students are asked to reflect on the solution and to highlight the main tool used to solve the problem. By the 5th and 6th problems the students are working comfortably with modulo arithmetic.

Activity:

1. Throughout p denotes a prime number;
2. Show that if $p > 2$ then there is a $k \in \mathbb{Z}$ such that $p = 2k + 1$;
3. Reformulate your previous answer in modulo arithmetic with respect to 2 (vertical mathematizing);
4. Examine what can be said if you replace modulo 2 with modulo $3, 4, 5,$ etc. (generalising);
5. Show that if $p > 3$ then $p = 6k \pm 1$ for some $k \in \mathbb{Z}$ (specialising);
6. Show that if $p > 5$ then $p^2 \equiv 1 \; mod \; 8$ (vertical mathematizing);
7. Deduce that if p, q are prime numbers greater than 5 then $p^2 - q^2$ is divisible by 24 (themed problem solving).

The solution to the 6th problem is an exemplar for the topic of this article. Having developed students' familiarity with vertical mathematizing in previous problems, they are then able to reformulate the 6th problem and solve it. The session concludes with a discussion of the intended outcomes: both in terms of the broader notion of themes in mathematics, and in terms of the specific theme of using modulo arithmetic to reformulate problems in number theory. Students are asked to reflect on other tools and themes they might have come across in other subjects.

6. Outcomes

This article is a case study of a second-year module that utilised the above approach to problem solving and no primary research has been done to determine its effectiveness. However, a number of positive outcomes have been seen on the programme as a whole. Having worked through a number of similar sessions involving activities such as the one described above, students demonstrate an increased confidence in approaching problems in areas of pure mathematics. Additionally, students develop clarity in their ability to reflect on proofs and solutions to problems that they see in their core modules.

By the time students see more complex and difficult proofs in their third year students are demonstrably able to break them down into their constituent themes and can highlight common techniques and tools from other areas.

Overall thematic problem solving is proving to be a success in its ability to de-mystify problem solutions and proofs in mathematics and it is expected that as we develop the range of activities in the future this will lead to a deeper understanding of core material.

7. References

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