Journal of Applied Non-Classical Logics
Vol. 00, No. 00, Month 201X, 1–32

RESEARCH ARTICLE

Game Theoretical Semantics for Some Non-Classical Logics

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(*Received 00 Month 201X; final version received 00 Month 201X*)

Paraconsistent logics are the formal systems in which absurduties do not trivialize the logic. In this paper, we give Hintikka-style game theoretical semantics for a variety of paraconsistent and non-classical logics. For this purpose, we consider Priest's Logic of Paradox, Dunn's First-Degree Entailment, Routleys' Relevant Logics, McCall's Connexive Logic and Belnap's Four-Valued Logic. We also present a game theoretical characterization of a translation between Logic of Paradox/Kleene's K3 and S5. We underline how non-classical logics require different verification games and prove the correctness theorems of their respective game theoretical semantics. This allows us to observe that paraconsistent logics break the classical bidirectional connection between winning strategies and truth values.
Keywords Game Theoretical Semantics; Logic of Paradox; First-Degree Entailment; Relevant Logic; Connexive Logic; Belnap's Four Valued Logic B4; Modal Logic S5.

1. Introduction

Game theoretical semantics suggests a very intuitive and natural approach to formal semantics and proofs. *The semantic verification game* for classical logic is played by two players, the *verifier* and the *falsifier*, who we call Heloise and Abelard respectively. The verifier's goal is to verify the truth of a given formula in a given model. Dually, the falsifier's goal is to falsify it. The rules of the semantic verification game are specified syntactically based on the form of the formula. During the play of the game, the given formula is broken into subformulas step by step by the players. The play of the game terminates when it reaches the propositional literals and when there is no move to make. If the play ends with a propositional literal which is true in the model in question, then the verifier wins the game. Otherwise, the falsifier wins. We associate conjunction with the falsifier, disjunction with the verifier. That is, when the main connective is a conjunction, it is the falsifier's turn to choose and make a move, and similarly, disjunction yields a choice for the verifier. The negation operator switches the roles of the players: the verifier becomes the falsifier and the falsifier becomes the verifier. Informally, a player is said to have a winning strategy if he has a set of rules that guides him throughout the play and tells him which move to make, and consequently gives him a win regardless of how the opponent plays. The major result of this approach states that Heloise the verifier has a winning strategy in the verification game if and only if the given formula is true in the given model. This is called the *correctness theorem* for game theoretical semantics for classical logic.

There can be found various foundational philosophical problems in game semantics. First, the verifier cannot be expected to verify a formula in a single play if she does not have a winning strategy. But the very existence of a winning strategy requires quantification over *plays*[1]. Second, a determined perfect information game such as a

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[1]We are grateful to one of the anonymous referees for pointing this out.

semantic verification game may be found *irrational* to play for the losing player. The reason is simple: computational complexity of verifying the truth of a formula (that is computing it from the truth table, in the classical propositional case) is very low. Then, it is reasonable to ask why a rational player would engage in such a simple determined perfect information game if she does not have a winning strategy? However important they are, such philosophical issues fall outside the scope of this work.

This paper is motivated by the fact that the semantic verification game and its rules are shaped by classical logic and consequently by its restrictions. Our goal is to observe how verification games change in some well-known propositional non-classical and paraconsistent logics when we strictly follow the Hintikkan methodology. In what follows, we obtain semantic verification games in which winning strategies for players are not sufficient conditions for establishing the truth values of the formulas, and observe further how the Hintikkan agenda may or may not be carried over to non-classical logics successfully.

A warning is in order here. Notice that in verification games players may change their roles throughout the game depending on the occurrences of the negation symbol[1]. Therefore, players' roles at the beginning of the game and at the end of the game may differ. Abelard may start the game as the falsifier but end up being the verifier. For this reason, the characterization theorems for game semantics and logics are given based on players' roles. This is easy when the game has two players with two immediate roles, such as the verifier and the falsifier. But, for multi-valued logics and non-classical logics, it is not easy to identify and name the precise roles of the players. Therefore, throughout the paper, we give the characterization theorems for players' roles and informally associate it with the players for clarity.

1.1 *Non-Classical Logics and Game Theoretical Semantics: A Brief Overview*

Game theoretical semantics (GTS, for short) was popularized by Jaakko Hintikka and Helsinki School researchers, even though similar ideas can be found in Parikh (much later published as (Parikh, 1999)).

Game semantics nowadays spans a very broad field attracting computer scientists and philosophers alike. Some extensive surveys of the subject can be found in (Abramsky & McCusker, 1999; A. Pietarinen & Sandu, 2000; A.-V. Pietarinen, 2003; Hintikka & Sandu, 1997). An overview of the field and its relation to various epistemic and scientific topics was discussed in (A.-V. Pietarinen, 2003).

The connection between non-classical logics and game semantics still remains understudied. A game theoretical reading of truth and its relation to winning strategies were investigated by (Boyer & Sandu, 2012). Pietarinen considered various non-classical issues including partiality and non-competitive games within the framework of GTS with some connections to the Kleene logic without focusing on particular (paraconsistent) logics (A. Pietarinen, 2000; A.-V. Pietarinen, 2004; Sandu & Pietarinen, 2001). Hintikka and Sandu discussed non-classicality in GTS also without specifically offering any insight on paraconsistency (Hintikka & Sandu, 1997; A. Pietarinen & Sandu, 2000). Tulenheimo studied languages with two negation signs, which can bear some resemblence to paraconsistent ideas on weak and strong negations (Tulenheimo, 2014). Additionally,

[1]A quick solution to this problem is to convert the given formula to its negation normal form with all the occurrences of negation symbols in front of the formula. This allows to distribute the roles at the very beginning of the play and prevents any further role redistribution during the play. However, for the most general case, we do not resort to this *trick*. Additionally, for many non-classical logics, failure of the De Morgan laws makes it difficult, if not impossible, to form the negation normal forms.

relating GTS to constructivism and intuitionism with some reference to type-theoretical foundations was presented by Ranta (Ranta, 1988). An epistemic, first-order extension of GTS, called "Independence-Friendly" logic, was suggested by Hintikka and Sandu (Hintikka & Sandu, 1989; Mann, Sandu, & Sevenster, 2011). Independence-friendly logics can be viewed as the logics for imperfect information games and they introduce Henkin quantifiers to game theoretical discussions.

It is worth noting how intuitionism can be approached from a GTS point of view. Hintikka remarked that the law of excluded middle may not hold in some instances of GTS. The reason is that the lack of a winning strategy for a player does not entail the existence of a winning strategy for the other player, which suggests that some semantic games may not be zero-sum (Hintikka, 1996). However, Hintikka himself was never very clear on GTS and intuitionism, especially when it came to negation (Tennant, 1998).

In this work, we consider a variety of well-known non-classical and paraconsistent logics. We define paraconsistent logic as any formal system that does *not* satisfy the explosion principle: $\varphi, \neg\varphi \vdash \psi$ for all φ, ψ. In other words, in paraconsistent logic, inconsistencies do not trivialize the system. There exists a wide variety of paraconsistent logics, and there are numerous ways to construct them (da Costa, Krause, & Bueno, 2007; Priest, 2002, 2007). Apart from its proof-theoretical definition, paraconsistency can also be described semantically suggesting that in paraconsistent logics some formulas and their negations can both be true[1]. In what follows, we will trace the reflections of this semantical condition on verification games, and ask how verification games might change when the logic allows non-trivial inconsistencies.

Apart from studying the underlying logic, GTS can also be approached from a purely game theoretical perspective. In the classical case, the verification games are constructed as zero-sum (a win for a player is a loss for the other), two-player, determined (one player always has a winning strategy), sequential (players do not make moves at the same time) non-cooperative games. It is then worthwhile to consider verification games where *i*) Abelard and Heloise both may win, *ii*) Abelard and Heloise both may lose, *iii*) Heloise may win, Abelard may not lose, *iv*) Abelard may win, Heloise may not lose, *v*) There is a tie, *vi*) There is an additional player, *vii*) Players play simultaneously, and *viii*) Players may cooperate. Such different possibilities can occur, for instance, when both p and $\neg p$ are true, so that both players can be expected to have winning strategies. We can also imagine verification games with additional truth values and additional players beyond verifiers and falsifiers, and also construct games where players may play concurrently in a parallel fashion.

Additionally, in the classical case, the existence of winning strategies and the truth values of formulas are closely connected. In particular, can players have winning strategies that cannot determine the truth value of the formula? Can the truth value of a formula be established if more than one player has a winning strategy? In what follows, we observe such *deviances* from the classical case.

Another motivation to approach GTS from a non-classical logical perspective is *game compositionality*. Combining logics using various game theoretical, semantical and model theoretical arguments is central in proof theory (Abramsky, 2007; Gabbay, Kurucz, Wolter, & Zakharyaschev, 2003). However, such work has not been extended to paraconsistent logics to the best of our knowledge. In this regard, constructing a theory for composing semantic games for different non-classical logics is still an open problem, and this work attempts to take a modest step towards that direction.

[1]Paraconsistency is usually given a proof-theoretical definition, as we did. Semantical (and metaphysical) commitment for the possibility of having true contradictions is called *dialetheism*. In this work, to prevent an inflation of terminology, we use the term "paraconsistency" both in proof-theoretical and semantical senses.

1.2 *Negation: Game Theoretical Motivation*

The problems of negation constitute one of the main motivations in the development of various non-classical logics. For our purposes, negation becomes problematic especially when it is considered within a game theoretical context. In the classical case, as we mentioned, the negation operator forces players to assume their opponent's role. Yet, in the non-classical case, it is not obvious how the new roles are determined. The following simple example illustrates our point.

Example 1.1. *Two men want to marry a princess. The king says they have to race on a horseback. The slowest one wins, and can marry the princess. How can one win this game and marry the princess?*

The solution simply suggests that the men need to swap their horses. Since the fastest one loses, and players race with each other's horse, what they need to do is to become the fastest in the dual game. The fastest one on the switched horse in the dual game wins the original game.

In the above example, it becomes clear how GTS for negation operates. If the slowest one wins the game G, then the fastest one wins the dual game G^d. There is certainly some sense of rationality here. That is, the players consider it easier in some sense to switch horses and race in the dual game. Yet, the above example and the idea are not strong enough to generalize. One of the most obvious difficulties arises when the same game is considered with 3 or more players. For $n > 2$ players, the solution requires a different understanding of negation. The similar complexity also carries over to binary connectives perceived as choice functions for certain players (Olde Loohuis & Venema, 2010).

For example, let us consider Example 1.1 with three players. If three players are supposed to switch horses for the dual game, it is possible that some players may end up with their own horses. The players who end up with their own horses can be viewed as the *fixed-points* for negation. Therefore, in some games, negation may not change the roles of some players and this needs to be addressed in verification games.

This paper aims at filling the gap in the literature between game theoretical semantics and paraconsistent non-classical logics. In what follows, we consider a variety of well-known paraconsistent logics, offer Hintikkan game semantics for them and observe how different logics generate different verification games. By this, we remark how the Hintikkan methodology applies to non-classical logics when the verification games and the players' roles are defined in the standard way. This project is philosophically important when winning strategies are seen as constructive proofs for truth in an intuitionistic sense or as verifications (Boyer & Sandu, 2012). Therefore, by focusing on inconsistent formulas and associated winning strategies, we offer alternative (constructive) proofs for inconsistencies and contribute to the computational discussions on the connection between proofs, strategies and truth. Furthermore, game semantics is a *non*-compositional semantics, where the truth value of a formula does not necessarily depend on the truth value of all its subformulas. Therefore, by constructing semantics games, this work suggests non-compositional semantics for various non-classical logics.

In this work, we consider Priest's Logic of Paradox, Dunn's First-Degree Entailment, Routleys' Relevant Logic, McColl's Connexive Logic, Belnap's Four Valued Logic and modal S5 and discuss their game semantics. *En passant*, we briefly discuss the Brazilian and Canadian schools of paraconsistency.

2. Semantic Games

As we argued earlier, semantic verification games for non-classical logics do not necessarily have the limitations of semantic games for classical logic which are two-player, zero-sum, non-cooperative, sequential and determined. For that reason, we define semantic games broadly to allocate the *deviances* of non-classical logics within game theoretical framework.

Let us now formally define GTS following the terminology given in (A. Pietarinen & Sandu, 2000). First, we take the language $\mathcal{L}$ of propositional logic with the following syntax for a set of countable propositional variables $\mathbf{P}$:

$$\varphi := \; p \;\mid\; \neg\varphi \;\mid\; \varphi \wedge \varphi \;\mid\; \varphi \vee \varphi$$

where $p \in \mathbf{P}$. The non-classical logics we consider here may or may not enjoy the De Morgan Laws for the interdefinability of the connectives. This, however, will not be our main concern here. In this respect, the above syntax includes the connectives that we believe are central in our non-classical inquiry of game semantics, and for simplicity, we assume that $\mathcal{L}$ has neither $\rightarrow$ nor $\leftrightarrow$. Throughout the paper, except when we consider modal S5, we assume the standard language $\mathcal{L}$ of propositional logic as defined above. Let us now define models.

Definition 2.1. A model M is a tuple (S, v) where S is a non-empty domain on which the game is played, and the valuation function v assigns the formulas in $\mathcal{L}$ to truth values in the logic.

In order to define a verification game in a model, we need some game theoretical components. First, we need a set of players with the supposition that each player forces a truth value. This defines players' goals, that is, they try to reach an atom that has the truth value that they are forcing. However, their roles can be changed in the play by the negation operator: a player may start the play with a specific role and end up with some other role. Second, we need rules telling each player what moves are possible at each stage depending on their roles. Sometimes, some players may make simultaneous moves, sometimes, some players may make no move at all. Since moves are made between game positions, we need to define them as well. A *game-token* can be used to indicate the current position of the players. Finally, it is possible to have concurrent or parallel play, and this needs to be specified. An inclusive definition of a verification game is given as follows.

Definition 2.2. A verification game is a tuple $\Gamma = (\pi, \rho, \sigma, \tau, \delta)$ where π is the set of players aiming at winning the play by reaching atomic formulas with specific truth values based on their roles, ρ is the set of well-defined game rules, σ is the set of positions, τ is the set of positions of the game-token in the case of a concurrent play, and δ is the set of designated truth values.

A more detailed explanation for the components of Γ is in order. The set of positions or game states σ is determined by the subformulas of the given formula and the players. We also embed the turn function at the positions into game rules for simplicity. Therefore, the set of positions will be composed of tuples as (p_i, φ) for $p_i \in \pi$ and a well defined formula φ. The tuple (p_i, φ) will read "it is player p_i's turn at φ". The set σ_{p_i} will denote the set of positions for player $p_i \in \pi$, and will be defined as $\sigma_{p_i} = \{(p_i, \varphi) : (p_i, \varphi) \in \sigma \text{ for a fixed player } p_i\}$.

The set σ is not sufficient by itself to describe concurrency in verification games as we also need to know which positions are tied together and played simultaneously. In the classical case, τ is the set of singletons of the form $\{(p_i, \varphi)\}$ for $p_i \in \pi$ and a well-defined formula φ as there is no parallel play. However, in non-classical verification games, as we may need to resort to concurrency, τ need not be composed of singletons. Therefore, $\{(p_1, \varphi), (p_2, \psi)\} \in \tau$ will then read that "player p_1 plays φ and player p_2 plays ψ concurrently (simultaneously)". For simplicity, we will not include the end-points (that is, the atoms) in τ. Additionally, in the logics we consider, only the binary connectives have the potential to create concurrency and parallel plays in the game. Therefore in the play, the current place of the game token is determined by σ and τ together. Moreover, the separation of σ and τ serves another purpose that the verification games in non-classical logics are not assumed to be zero-sum. Thus, the outcome of one subgame does not necessarily determine the outcome of its parallel play, if it exists.

Given two positions (p_i, φ) and (p_j, ψ) for $i \neq j \in \pi$, we say that they are played concurrently (or in parallel) if $\exists X \in \tau$ such that $X = \{(p_i, \varphi), (p_j, \psi)\}$. The definition can easily be extended to n-concurrency where n different players are allowed to play concurrently. Throughout this work, we will use "concurrency" and "parallel play" interchangeably for easy reading.

It is important to see that this reading of concurrency in games entails that concurrent moves result in two separate positions. This separation is essential as it does not presuppose any additional game theoretical restrictions and keeps the model simple. Clearly, it is possible to define game theoretical concurrency for different purposes, yet we stick to our definition for simplicity.

The set of game rules ρ will be defined inductively as transformations from a game position (p_i, φ) to a set of game positions $\{(p_j, \psi)\}_{j \in I}$ for $p_i \in \pi$, $I \subseteq \pi$, well-defined formula φ and a subformula ψ of φ, defined in the standard way. For a rule $r \in \rho$ and a set of positions O, the set $r(O)$ denotes the set of positions obtained from applying r to the set of positions in O (where applicable). When specifying the set ρ for the logics we discuss here, we will use the informal descriptions of the rules for simplicity. It will also help us to maintain the Hintikkan intuition about game semantics.

Finally, the set of designated truth values are used to define theorems in a given particular logic. They can be viewed as a non-classical extension of the truth value True, and are preserved under valid inferences. For example, two logics may enjoy the same truth table but can be distinguished by a different set of designated truth values.

Game semantics naturally introduces game theoretical concepts to logic. In this respect, a *strategy* for a player is a set of rules that tells him which move to make at each position where it is his turn. A *winning strategy* is a strategy that guarantees a win for the player regardless of the moves of the opponents.

The play of a verification game terminates when it reaches atomic formulas, and the winner is determined by the truth value of the atom. In the classical case for example, if we end up with an atom with a truth value T, the verifier wins the game[1].

In the non-classical case, the existence of a winning strategy for a player does not necessarily entail the non-existence of winning strategies for the opponents. Moreover, some players may force some truth values jointly (by forming coalitions, for example) or some players may force no truth values (which can be called *noneist* players or *nullifiers*). In this work, we focus on the existence of winning strategies and how this relates to the truth value of the formula in question. For this reason, strategies (or winning strategies) themselves are not the main focus of this work.

Formally, a strategy $s(p_i)$ for a player p_i is a function from a set of positions to the set of positions obtained by applying the particular rules in ρ. Formally, we define $s_{p_i} : \sigma_{p_i} \mapsto \bigcup_{r \in \rho} r(\sigma_{p_i})$ where $p_i \in \pi$, σ_{p_i} is the set of positions for player p_i, and $r(\sigma_{p_i})$ is the set of positions for p_i obtained after the application of rule $r \in \rho$. Notice that

[1]Notice again that the initial and final configurations of the roles and the players may differ. For instance, in some plays, Abelard may start as the falsifier yet may finish as the verifier.

$r(\sigma_{p_i})$ is not necessarily a subset of σ_{p_i}. Also, it is important to observe that strategies are defined from a set of positions, not from a set of game tokens. The reason is that when players play concurrently each may have their own strategy independent from each other.

Finally, we denote the *semantic verification game* Γ for a model M and a well-defined formula $\varphi \in \mathcal{L}$ by $\Gamma(M, \varphi)$.

The framework we have presented so far has the expressive power for the semantic games we consider in this work. But more importantly, it carries the Hintikkan game theoretical intuition over to non-classical logical semantic games, which is one of our main focal points in this work.

3. Game Semantics for Logic of Paradox

Logic of paradox (LP, for short) introduces an additional truth value P, called *paradoxical*, which intuitively stands for both true and false (Priest, 1979).

The logics LP and Kleene's three valued logic K3 have the same truth tables. However, they differ on the truth values that they preserve in valid inferences, and how they read P. The truth values that are preserved in validities are called *designated truth values* and they can be thought of as the extensions of the classical notion of truth (Priest, 2008). In LP, it is the set $\{T, P\}$; in K3 (and classical logic), it is the set $\{T\}$. Even if the truth tables of two logics are the same, different sets of designated truth values produce different sets of validities, thus different logics. For instance, $p \vee \neg p$ is a theorem in LP, but not in K3. In K3, the third truth value has an intuitionistic reading and can be viewed as an undervaluation in contrast to its reading as an overvaluation in LP. It is also important to note that the set of validities of LP contains the set of validities of the classical logic.

	$\neg$
T	F
F	T
P	P

$\wedge$	T	P	F
T	T	P	F
P	P	P	F
F	F	F	F

$\vee$	T	P	F
T	T	T	T
P	T	P	P
F	T	P	F

Figure 1. *The truth tables for LP and K3.*

We stipulate that the introduction of the third truth value requires an additional player that we call *Astrolabe*[1]. Astrolabe is the *paradoxifier* in the game forcing the game to an end with the truth value P.

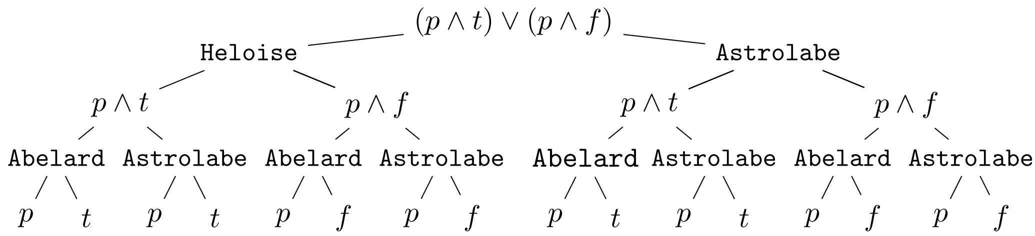
In GTS for LP, the first problem is to determine the turns and the moves of three players at each connective. For instance, the problem surfaces clearly when the formula $t \wedge p$, where t and p have the truth values T and P respectively, is considered. In this quick game, if we assume that it is Abelard the falsifier's turn then he will not have a move that can give him a win. From the truth table, it can be seen that the formula evaluates to P, therefore Astrolabe the paradoxifier can be expected to have a winning strategy. In order to make it possible, Astrolabe must be allowed to make moves at conjunctions. Similarly, let us consider $f \vee p$, where f and p have the truth values F and P respectively. As it can be computed from the LP truth table, the formula $f \vee p$ has the truth value P. Now, Heloise cannot make a move that can give her a win. Also, Astrolabe needs to be given a turn to make a move to win. Therefore, we associate disjunction with Heloise and Astrolabe, and conjunction with Abelard and Astrolabe. This modification introduces parallel play where the players may make moves in a

[1]Astrolabe was the name of Abelard and Heloise's son.

parallel, concurrent fashion independently. In the case of negation, Heloise and Abelard will switch their role, and Astrolabe will keep his role as P is a fixed-point for negation in LP. Astrolabe's role always remains as the paradoxifier in the game.

The following example will be helpful in determining the rules for LP semantic games. For simplicity and easiness in dealing with the non-classical truth values, we will use the following convention: the proposition variable p will have a truth value P, t will have T and f will have F and so on.

Example 3.1. Consider the formula $(p \wedge t) \vee (p \wedge f)$ which evaluates to P in LP. In this game, Astrolabe the paradoxifier has a winning strategy: at each end-node ($p \wedge t$ and $p \wedge f$), he selects p. Here, we also observe that Abelard being stuck at some states (such as $p \wedge t$) does not necessarily entail a win for the other players.



For this illustrative game, the game token set τ contains, for example, the tuples $\{(\text{Heloise}, (p \wedge t) \vee (p \wedge f)), (\text{Astrolabe}, (p \wedge t) \vee (p \wedge f))\}$ and $\{(\text{Abelard}, p \wedge t), (\text{Astrolabe}, p \wedge t)\}$ which read that players Heloise and Astrolabe play simultaneously at $(p \wedge t) \vee (p \wedge f)$, and players Abelard and Astrolabe play at $p \wedge t$ simultaneously, respectively.

We denote the verification game for LP by GTS^{LP}. GTS^{LP} is a non-zero sum verification game where more than one player may have a winning strategy. Since GTS^{LP} is non-zero sum, making the opponent lose does not necessarily entail that it is a win for the player himself. As we will observe later on, in GTS^{LP} admitting winning strategies does not necessarily entail the truth value of the formula in question.

Definition 3.2. The tuple $\Gamma_{\text{LP}} = (\pi, \rho, \sigma, \tau, \delta)$ is a verification game for LP where $\pi = \{\text{Astrolabe, Heloise, Abelard}\}$, σ is the set of tuples (p_i, φ) for $p_i \in \pi$ and a well-formed formula φ, and δ is $\{T, P\}$. For a game $\Gamma_{\text{LP}}(M, \varphi)$, τ is given inductively for the positions (p_i, φ) in σ as follows,

- if $\varphi = \neg\psi$, then,
  - if $\{(\text{Abelard}, \neg\psi)\} \in \tau$ then, $\{(\text{Heloise}, \psi)\} \in \tau$,
  - if $\{(\text{Heloise}, \neg\psi)\} \in \tau$ then, $\{(\text{Abelard}, \psi)\} \in \tau$,
  - if $\{(\text{Astrolabe}, \neg\psi)\} \in \tau$ then, $\{(\text{Astrolabe}, \psi)\} \in \tau$,
- if $\varphi = \chi \wedge \psi$, then $\{(\text{Abelard}, \chi \wedge \psi), (\text{Astrolabe}, \chi \wedge \psi)\} \in \tau$,
- if $\varphi = \chi \vee \psi$, then $\{(\text{Heloise}, \chi \vee \psi), (\text{Astrolabe}, \chi \vee \psi)\} \in \tau$;

and, finally, ρ is given inductively as follows.

- If φ is atomic, the game terminates, and Heloise wins if φ is true, Abelard wins if φ is false and Astrolabe wins if φ is paradoxical,
- if $\varphi = \neg\psi$, Abelard and Heloise switch roles, Astrolabe keeps his role, and the game continues as $\Gamma_{\text{LP}}(M, \psi)$,
- if $\varphi = \chi \wedge \psi$, Abelard and Astrolabe choose between χ and ψ simultaneously,
- if $\varphi = \chi \vee \psi$, Heloise and Astrolabe choose between χ and ψ simultaneously.

As we mentioned earlier, game rules ρ is given informally above. They can be made more precise. For example, the last game rule for disjunction can formally be written down as a function r as follows

$$r(p_i, \chi \vee \psi) = \begin{cases} \{(p_j, \chi), (p_k, \psi)\} & \text{if } p_i \text{ is Heloise,} \\ \{(p_j, \chi), (p_k, \psi)\} & \text{if } p_i \text{ is Astrolabe,} \end{cases}$$

where $p_i, p_j, p_k \in \pi$, $\{(\text{Heloise}, \chi \vee \psi), (\text{Astrolabe}, \chi \vee \psi)\} \in \tau$, and game continues as either $\Gamma_{\text{LP}}(M, \chi)$ or $\Gamma_{\text{LP}}(M, \psi)$ with whomever player is supposed to make the next move. However, in order to reflect the game theoretical intuition more clearly, from now on, we will specify the game rules as in Definition 3.2. Additionally, in order to economize in the formal game definitions, from now on we will define τ only for binary connectives as negation does not create a parallel play.

As we have noted before, the correctness theorems are given with respect to players' roles. In order to avoid any possible confusion, we will put for example, "Heloise the verifier" to specify that the theorem applies to the player who is the verifier - even if it is not Heloise. This is for easy read, especially when we have non-classical players with non-classical roles, such as paradoxifier.

Theorem 3.3. *In a GTS^{LP} verification game $\Gamma_{\text{LP}}(M, \varphi)$,*

- *Heloise the verifier has a winning strategy if φ is true in M,*
- *Abelard the falsifier has a winning strategy if φ is false in M,*
- *Astrolabe the pardoxifier has a winning strategy if φ is paradoxical in M.*

Proof. We start with the case for Heloise the verifier. Let us and consider the game state (Heloise, φ). We proceed by induction on the complexity of φ and describe Heloise the verifier's winning strategy for each case. Now, let φ be true in M.

If φ is a propositional letter p which is true in M, then Heloise the verifier wins by definition, hence has a winning strategy.

Let $\varphi = \neg\psi$. Then, ψ is false. By the game rules, the play continues where Heloise is the falsifier. By the induction hypothesis (for falsifier), Heloise the falsifier has a winning strategy for ψ. Then, she as the verifier has a winning strategy for $\neg\psi$ by simply playing her game as the falsifier for ψ. Thus, she has a winning strategy for φ.

Now, let φ be a conjunction of the form $\chi \wedge \psi$. Since, φ is assumed to be true, the only way to make it true is to have χ and ψ both true. Then, by the induction hypothesis, Heloise the verifier has a winning strategy for both χ and ψ. Then, for φ, Abelard the falsifier and Astrolabe the paradoxifier make moves. Yet, whichever move they make (whichever of χ or ψ they choose), Heloise the verifier will have a winning strategy. Thus, for φ, she has a winning strategy: whatever move Abelard and Astrolabe make, she wins.

Let φ be a disjunction of the form $\chi \vee \psi$. Then, by the induction hypothesis, Heloise has a winning strategy for either χ or ψ, whichever is true. Then, choosing the true disjunct is her winning strategy at φ, independent from whatever Astrolabe chooses.

The case for Abelard the falsifier, that is, the game positions (Abelard, φ), is almost identical to that of Heloise's, hence skipped.

For Astrolabe the paradoxifier, we first assume that the given formula φ is paradoxical. Let φ be paradoxical in M. Consider the game state (Astrolabe, φ)

If φ is a propositional letter p which is paradoxical in M, then Astrolabe wins the play by definition, hence has a winning strategy.

Let $\varphi = \neg\psi$. Then, ψ is paradoxical, too. By the game rules, Astrolabe the paradoxifier's rule remains the same. By the induction hypothesis, he has a winning strategy for ψ. Then, he has a winning strategy for φ by simply maintaining the same role and the strategy, and proceeding with ψ.

For $\varphi = \chi \wedge \psi$. Since φ is assumed to be paradoxical, we only have two options for

χ and ψ: (1) either one of them has the truth value P and the other has the truth value T, (2) both have the truth value P. Therefore, Astrolabe the paradoxifier has a winning strategy for at least one of χ and ψ, by the induction hypothesis. Then, for φ, Astrolabe chooses the conjunct that has the truth value P for which he has a winning strategy already. This forms his winning strategy for φ, independent from whatever move Abelard the falsifier makes.

If $\varphi = \chi \vee \psi$, then we have two options as well: (1) one of the disjuncts has the truth value P and the other one has the truth value F, (2) both have the truth value P. So, Astrolabe the paradoxifier has a winning strategy for at least one of χ and ψ, by the induction hypothesis. Then, for the game with φ, Astrolabe chooses the disjunct that has the truth value P, and this forms his winning strategy for φ, independent from whatever Heloise chooses. ■

Theorem 3.3 is what is called *the correctness theorem* for GTS$^{\text{LP}}$ which establishes how GTS captures the standard semantics of LP. We will refer to such theorems with this name from now on.

In the light of Theorem 3.3, it is important to note that LP distinguishes different trues and falses: trues that are only true (T), falses that are only false (F), and trues that are also false (P) and falses that are also true (P). In GTS, this distinction carries over to games allowing Astrolabe the paradoxifier to make moves alongside Heloise the verifier and Abelard the falsifier. The purpose of this extended set of game rules in LP is to cover the possibility of trues that are also false and falses that are also true. In GTS$^{\text{LP}}$, there are winning strategies which cause a loss for the opponent, and there are winning strategies which do not. Additionally, there are winning strategies that cannot guarantee the logical truth of the formula in GTS$^{\text{LP}}$. A play for $p \wedge f$ illustrates this point, where both Abelard and Astrolabe can have a winning strategy. But this does not directly say anything about determining the truth value of $p \wedge f$. Therefore, in GTS$^{\text{LP}}$, the immediate connection between the existence of winning strategies and truth values becomes slightly more complicated as the following theorem identifies.

Theorem 3.4. *In a GTS^{LP} verification game* $\Gamma_{\text{LP}}(M, \varphi)$,

- *If Heloise the verifier has a winning strategy, then φ is true in M,*
- *If Abelard the falsifier has a winning strategy, then φ is false in M,*
- *If Astrolabe the paradoxifier has a winning strategy, but not the other players, then φ is paradoxical in M.*

Proof. We start with the case for Heloise the verifier. Assume that for φ Heloise the verifier has a winning strategy at state (Heloise, φ). The proof is by induction on φ.

If φ is a propositional variable p for which Heloise the verifier has a winning strategy, then p is true by definition.

If $\varphi = \neg\psi$, then by definition, Heloise the falsifier has a winning strategy for ψ. By the induction hypothesis, then ψ as false. Then, $\neg\psi$, that is, φ is true.

Let $\varphi = \chi \wedge \psi$. If Heloise the verifier has a winning strategy, then she must have winning strategies for both χ and ψ as it is not her turn. By the induction hypothesis both χ and ψ must be true. In this case, φ turns out to be true as well.

Let $\varphi = \chi \vee \psi$. If Heloise the verifier has a winning strategy, this means that Heloise has a winning strategy for at least one of χ and ψ. Then, by the induction hypothesis one of them is true. By the truth table for LP, this makes φ true as well. Notice that it is also Astrolabe the paradoxifier's turn to choose here. If he has a winning strategy, he can select the disjunct that gives him a win. By the induction hypothesis, that disjunct can be paradoxical. Yet, as the other conjunct is true (which is by assumption as Heloise has a winning strategy), φ still becomes true (as $T \vee P$ is true).

The case for Abelard the falsifier is very similar, hence skipped.

The interesting case is that of Astrolabe the paradoxifier. Assume that for φ only Astrolabe the paradoxifier has a winning strategy at position (Astrolabe, φ).

If φ is a propositional variable, and Astrolabe has a winning strategy, then by definition φ is paradoxical.

If $\varphi = \neg\psi$, and only Astrolabe has a winning strategy, he keeps playing for ψ as a paradoxifier. By the induction hypothesis, this means that ψ is paradoxical. By the truth table, φ becomes paradoxical as well.

Now, let $\varphi = \chi \wedge \psi$. If only Astrolabe the paradoxifier has a winning strategy, this means that he has a winning strategy for either of the conjuncts (as he can choose whichever he likes), say χ without loss of generality. Then, by the induction hypothesis, χ is paradoxical. Since Abelard does not have a winning strategy, by Theorem 3.3, then neither of the conjuncts is false. Thus, by the truth table, φ is forced to be paradoxical as χ is paradoxical. Otherwise, if Abelard had a winning strategy, and if one of the conjuncts was F, then $P \wedge F$ would return F, not P disproving the claim. This is the reason why only Astrolabe is supposed to have a winning strategy.

Finally, if $\varphi = \chi \vee \psi$, then if only Astrolabe the paradoxifier has a winning strategy, then he has a winning strategy for either of the disjuncts, say χ without loss of generality. Then, by the induction hypothesis, χ is paradoxical. Heloise not having a winning strategy means that neither of the disjuncts is true by Theorem 3.3. Then by the truth table, φ is forced to be paradoxical as χ is paradoxical. ■

It is important to note that the above theorems also show that verification games $\Gamma_{\text{LP}}(M, \varphi)$ for LP are determined as valuations in game models are functional (that is, each and every formula has exactly one truth value) and for each game there is at least one player with a winning strategy.

Theorem 3.4 also indicates that Astrolabe the paradoxifier's strategy is strictly dominated in a sense that if some other player also admits a winning strategy, then Astrolabe's strategy will not give him a win. This observation is another reading of the truth table for LP given in Figure 1.

Now, as a thought-experiment, it is possible to change some game rules in order to give a biconditional correctness theorem for GTS^{LP} by prioritizing some players over others. This will allow some players to dominate the others reflecting the truth table for LP. In this new and extended reading of GTS^{LP}, such a *move priority* is given to the *parents* (Abelard and Heloise). They are allowed to play first, then Astrolabe makes his move. This extension prevents parallel moves and incorporates winning strategies into the game rules. The revised game rules are given as follows.

- For propositional letters and negation, the rules are as before.
- Disjunction belongs to Heloise and Astrolabe; conjunction belongs to Abelard and Astrolabe.
- If Heloise (resp. Abelard) has a winning strategy in the sub-game they choose, the game proceeds with her (resp. his) move.
- Otherwise, Astrolabe makes a move.

The following example illustrates this modification.

Example 3.5. Let us consider the formula in Example 3.1. Given $(p \wedge t) \vee (p \wedge f)$, Heloise first attempts to choose either of them only to realize that she does not have a winning strategy in either of the sub-games with $p \wedge t$ or $p \wedge f$. So, she cannot make a move. Then, it becomes Astrolabe's turn. Astrolabe chooses $p \wedge t$. Now, Abelard attempts to choose either p or t only to realize that neither gives him a win. So, he cannot make a move. Astrolabe makes a move, chooses p, and wins - this is Astrolabe's winning strategy. If

Astrolabe chose $p \wedge f$, then first Abelard would make a move and choose f for a win. Yet, Abelard still does not have winning strategy in this game.

As we mentioned earlier, such a twist on $\mathrm{GTS}^{\mathrm{LP}}$ to recover the biconditionality in the correctness theorem is *ad-hoc*. It incorporates the existence of winning strategies into game rules creating a circular reasoning which results in the loss of game theoretical intuition. This is a big price to pay to have a biconditional version of Theorem 3.3. We shall observe a similar problem in due course for some other logics.

For another attempt to obtain a biconditional and classical correctness theorem, it can be argued that the introduction of the third player is arbitrary, and his role can be played by both players allowing them to play simultaneously. This line of argument also suggests that Astrolabe's win in a verification game is identical to having both Abelard and Heloise win the play at the same time.

The idea of covering the third truth value by some combination of the original two truth values is not strong enough to generalize to multi-valued logics. The situation gets more complicated when 4-valued logics are considered. More importantly, from a game theoretical perspective, identifying players' roles (whether they are verifiers, falsifiers, paradoxifiers or else) is an important game theoretical conceptualization in formal semantics (Hodges, 2013). Reducing some truth values to some combinations of others in an absolute reductionist sense eliminates the game theoretical sophistication and richness of verification games. A play may end with the same outcome when it is played by 2 or 4 players, but it does not suggest that they are indeed the same game. The results may be identical but the process, the play and more importantly the game theoretical and rational interaction among the players will not be identical. Such an agenda will inevitably fail if we want to explore the full richness of GTS for non-classical logics together with its logical and game theoretical implications.

In this section, we only discussed the basic system of LP. Kleene's K3, as we mentioned, admits the same truth table and the above GTS works for K3 with some small modifications. Additionally, some extensions of LP, such as RM3 which admits modus ponens, or minimal LP can also be given a GTS. We leave such extensions of LP to future work.

4. Game Semantics for First-Degree Entailment

Semantic evaluations are generally thought of as *functions* from logical formulas to truth values. This ensures that each and every formula is assigned a *unique* truth value. However, it is possible to replace the valuation function with a valuation *relation* which can produce multiple or no truth values for logical formulas. The system obtained in this manner is called *First-degree entailment* (FDE, for short), and is due to Dunn (Anderson & Belnap, 1963; Dunn, 1976).

For the given propositional language $\mathcal{L}$, the valuation relation $\mathbf{r}$ is defined on $\mathcal{L} \times \{0, 1\}$. By $\varphi \mathbf{r} \emptyset$, we denote the situation where φ is not related to any truth value, and (with a slight abuse of notation) by $\varphi \mathbf{r} \{0, 1\}$ the situation where it is related to both truth values. FDE is a paraconsistent (inconsistency-tolerant) and paracomplete (incompleteness-tolerant) logic. For formulas $\varphi, \psi \in \mathcal{L}$, the valuation $\mathbf{r}$ is defined inductively as follows.

- $\neg\varphi \mathbf{r} 1$ iff $\varphi \mathbf{r} 0$,
- $\neg\varphi \mathbf{r} 0$ iff $\varphi \mathbf{r} 1$,
- $(\varphi \wedge \psi) \mathbf{r} 1$ iff $\varphi \mathbf{r} 1$ and $\psi \mathbf{r} 1$,
- $(\varphi \wedge \psi) \mathbf{r} 0$ iff $\varphi \mathbf{r} 0$ or $\psi \mathbf{r} 0$,
- $(\varphi \vee \psi) \mathbf{r} 1$ iff $\varphi \mathbf{r} 1$ or $\psi \mathbf{r} 1$,
- $(\varphi \vee \psi) \mathbf{r} 0$ iff $\varphi \mathbf{r} 0$ and $\psi \mathbf{r} 0$.

Notice that LP can be obtained from FDE by imposing a restriction that no formula gets the truth value $\emptyset$.

What does the relational semantics correspond to in verification games? If the truth value P in LP can intuitively be thought of as both true and false, and if this allows concurrent moves in GTS$^{\text{LP}}$, then the same approach should work in game semantics for FDE as well. In FDE, unlike LP, formulas can have no truth values which suggests that neither Heloise nor Abelard may have a winning strategy (incompleteness-tolerance). Also, in FDE, both players can have winning strategies (inconsistency-tolerance). Since there exists a possibility that no players may have a winning strategy, semantic games for FDE are not determined.

We define the verification games for FDE in the standard fashion as follows where the GTS for FDE is denoted by GTS$^{\text{FDE}}$.

Definition 4.1. The tuple $\Gamma_{\text{FDE}} = (\pi, \rho, \sigma, \tau, \delta)$ is a verification game for FDE where $\pi = \{\text{Heloise, Abelard}\}$, σ is composed of tuples in the form (Heloise, φ) or (Abelard, φ) or $(\emptyset, \varphi)$ where $\emptyset$ denotes that no player has a turn at φ, and δ is $\{T\}$. For a game $\Gamma_{\text{FDE}}(M, \varphi)$, τ is given inductively for the positions (p_i, φ) in σ as follows,

- if $\varphi = \neg\psi$, then,
  - if $\{(\text{Abelard}, \neg\psi)\} \in \tau$ then, $\{(\text{Heloise}, \psi)\} \in \tau$,
  - if $\{(\text{Heloise}, \neg\psi)\} \in \tau$ then, $\{(\text{Abelard}, \psi)\} \in \tau$,
- if $\varphi = \chi \wedge \psi$, then $\{(\text{Heloise}, \chi \wedge \psi), (\text{Abelard}, \chi \wedge \psi)\} \in \tau$,
- if $\varphi = \chi \vee \psi$, then $\{(\text{Heloise}, \chi \vee \psi), (\text{Abelard}, \chi \vee \psi)\} \in \tau$;

and, finally, ρ is given inductively as follows.

- If φ is atomic, the game terminates, and Heloise wins if $\varphi\mathbf{r}1$, Abelard wins if $\varphi\mathbf{r}0$, neither wins if $\varphi\mathbf{r}\emptyset$,
- if $\varphi = \neg\psi$, players switch roles, and the game continues as $\Gamma_{\text{FDE}}(M, \psi)$,
- if $\varphi = \chi \wedge \psi$, Abelard and Heloise choose between χ and ψ simultaneously,
- if $\varphi = \chi \vee \psi$, Abelard and Heloise choose between χ and ψ simultaneously.

Game rules for GTS$^{\text{FDE}}$ also include the turn function which specifies that both players make moves at all binary connectives. A simple example can be helpful.

Example 4.2. Consider the formula $p \wedge (q \vee r)$ where $p\mathbf{r}\{0, 1\}$, $q\mathbf{r}\emptyset$ and $r\mathbf{r}0$. Then, the formula $p \wedge (q \vee r)$ evaluates to 0. In the verification game, Abelard the falsifier first chooses $q \vee r$, and then chooses r. Alternatively, he can also choose p as his winning strategy, yet this also cause Heloise the verifier to win the play.

This example is another case where the existence of winning strategies do not directly guarantee the truth value of the formula in question.

The correctness theorem for GTS$^{\text{FDE}}$ is given as follows.

Theorem 4.3. *In a game $\Gamma_{\text{FDE}}(M, \varphi)$, we have the following:*

- *Heloise the verifier has a winning strategy if $\varphi\mathbf{r}1$,*
- *Abelard the falsifier has a winning strategy if $\varphi\mathbf{r}0$,*
- *Either of the players or no player has a winning strategy if $\varphi\mathbf{r}\emptyset$.*

Proof. The proof is by induction for each case.

Case for $\varphi\mathbf{r}1$: We start with the case for Heloise the verifier at the game position (Heloise, φ). Suppose $\varphi\mathbf{r}1$. The cases for propositional variables and negation are immediate.

Let $\varphi = \chi \wedge \psi$. If $\varphi\mathbf{r}1$, then we have both $\chi\mathbf{r}1$ and $\psi\mathbf{r}1$. By the induction hypothesis, Heloise the verifier has winning strategies for both χ and ψ. Thus, she has a winning strategy for φ. For the failure of the reverse direction, assume that Heloise has a winning strategy. Without loss of generality, suppose that she chooses χ. Assume further that Abelard the falsifier has a winning strategy as well, that is, he chooses ψ. Then, by the induction hypothesis $\chi\mathbf{r}1$ and $\psi\mathbf{r}0$ which forces $\varphi\mathbf{r}0$.

Similarly, let $\varphi = \chi \vee \psi$. Then, we have $\chi\mathbf{r}1$ or $\psi\mathbf{r}1$. By the induction hypothesis, Heloise the verifier has a winning strategy for χ or ψ. Therefore, she has a winning strategy for φ, that is, she chooses the disjunct for which she has a winning strategy.

Case for $\varphi\mathbf{r}0$: The cases for Abelard the falsifier are very similar to the above, hence skipped.

Case for $\varphi\mathbf{r}\emptyset$: Consider the position $(\emptyset, \varphi)$ with $\varphi\mathbf{r}\emptyset$. If φ is a propositional variable, by definition, no player wins. Similarly, let $\varphi = \neg\psi$. Then, by definition, $\psi\mathbf{r}\emptyset$. By the induction hypothesis, no player has a winning strategy. Thus, no player has a winning strategy for φ.

Let $\varphi = \chi \wedge \psi$. Then, by the truth conditions for FDE, we have two options: (1) both $\chi\mathbf{r}\emptyset$ and $\psi\mathbf{r}\emptyset$, or (2) $\chi\mathbf{r}1$ and $\psi\mathbf{r}\emptyset$ (without loss of generality). If the prior is the case, by the induction hypothesis, no player has a winning strategy for χ or ψ. Thus, no player has a winning strategy for φ. If the latter is the case, then Heloise the verifier has a winning strategy for φ as he can make a move at a conjunction which forms his winning strategy for φ.

Dually, let $\varphi = \chi \vee \psi$. Then, by the truth conditions for FDE, we have two options: (1) both $\chi\mathbf{r}\emptyset$ and $\psi\mathbf{r}\emptyset$, or (2) $\chi\mathbf{r}0$ and $\psi\mathbf{r}\emptyset$ (without loss of generality). If the prior one is the case, by the same argument as above, no player has a winning strategy for φ. If the latter is the case, Astrolabe the falsifier chooses χ which constitutes her winning strategy for φ. ■

The above theorem suggests a different reading of determinacy. Even if games $\Gamma_{\text{FDE}}(M, \varphi)$ are finite and expected to be determined by the Gale-Stewart theorem, we observe that there can be cases where no player can have a winning strategy. Therefore, due to the underlying logic, some verification games for FDE can be undetermined. This immediately suggests a different reading of strategies as not functions, but as relations. We leave such extensions to future work.

Furthermore, if we impose that the $\Gamma_{\text{FDE}}(M, \varphi)$ games need to be determined, what we obtain is a $\Gamma_{\text{LP}}(M, \varphi)$ game. On the other hand, another reason why $\Gamma_{\text{FDE}}(M, \varphi)$ is not determined is because the game lacks a player whose role is to force the empty truth value $\emptyset$. It is a matter of philosophical debate whether $\emptyset$ is a truth value or whether it can be forced by a player. However, in the current framework, we chose not to introduce a player for that role. In due time, we will consider an extension of FDE that has four players: the verifier, the falsifier, the paradoxifier and the nullifier.

The connection between FDE and LP can further be explicated as follows.

Corollary 4.4. *For an LP model M and a formula φ, let M' be the model obtained from M by maintaining the same carrier set and replacing the valuation function of LP with the valuation relation of FDE as follows: $T \mapsto 1$, $F \mapsto 0$ and $P \mapsto \{0, 1\}$.*

If Heloise or Abelard has a winning strategy in $\Gamma_{\text{LP}}(M, \varphi)$, then Heloise or Abelard has a winning strategy in $\Gamma_{\text{FDE}}(M', \varphi)$ respectively. If only Astrolabe has a winning strategy in $\Gamma_{\text{LP}}(M, \varphi)$, then both Heloise and Abelard have winning strategies in $\Gamma_{\text{FDE}}(M', \varphi)$.

Proof. The first part about Heloise and Abelard follows from Theorem 3.4 and Theorem 4.3. In other words, if Heloise has a winning strategy in an LP game, then the formula is true in LP by Theorem 3.4. The translation then translates T of LP to 1 of FDE. Then, by Theorem 4.3, Heloise has a winning strategy in the FDE game. The argument is similar for Abelard.

If only Astrolabe has a winning strategy for the LP game for φ, then by Theorem 3.4,

φ is paradoxical. By the translation, then φ is related to both 0 and 1 in FDE. By Theorem 4.3, then both Heloise and Abelard have winning strategies in the FDE game. ∎

The converse of Corollary 4.4 is not true. In GTS$^{\text{FDE}}$, for $t \wedge f$, both Abelard and Heloise have winning strategies. Yet, in LP for a game $t \wedge f$, Heloise does not have a winning strategy.

5. Game Semantics for A Relevant Logic

Relevant logics define negation by resorting to possible worlds, and this reading renders negation as a modal operator. The idea is due to Routley and Routley (Routley & Routley, 1972). A *Routley model* is a structure $(W, \#, v)$ where W is a set of possible worlds, $\#$ is a map from W to itself, and v is a valuation function defined in the standard way. In this system, the semantics for disjunction and conjunction is local, whereas for negation, possible worlds are needed. The semantics is given as follows.

$$\begin{array}{lll} v(w, \neg\varphi) = 1 & \text{iff} & v(\#w, \varphi) = 0, \\ v(w, \varphi \wedge \psi) = 1 & \text{iff} & v(w, \varphi) = 1 \text{ and } v(w, \psi) = 1, \\ v(w, \varphi \vee \psi) = 1 & \text{iff} & v(w, \varphi) = 1 \text{ or } v(w, \psi) = 1. \end{array}$$

We call Routleys' system RR, and denote its GTS by GTS$^{\text{RR}}$. Notice that if $\#w = w$, then we have the classical truth conditions. Further connections between RR and FDE or LP can be found in (Priest, 2007). We denote semantical games in RR by $\Gamma_{\text{RR}}(M, \varphi, w)$ where M and φ are as before, and $w \in W$ is a possible world.

Definition 5.1. The tuple $\Gamma_{\text{RR}} = (\pi, \rho, \sigma, \tau, \delta)$ is a verification game for RR where $\pi = \{\text{Heloise, Abelard}\}$, σ is the set of game states in the form of (p_i, φ, w) for $p_i \in \pi$, $\varphi \in \mathcal{L}$ and $w \in W$, τ is composed of singleton game states as there is no parallel play, and δ is $\{T\}$. In a game $\Gamma_{\text{RR}}(M, \varphi, w)$, ρ is given as follows inductively for a fixed w.

- If φ is atomic, the game terminates, and Heloise wins if φ is true, Abelard wins if φ is false,
- if $\varphi = \neg\psi$, the players switch roles, and the game continues as $\Gamma_{\text{RR}}(M, \psi, \#w)$,
- if $\varphi = \chi \vee \psi$, Heloise the verifier chooses between χ and ψ,
- if $\varphi = \chi \wedge \psi$, Abelard the falsifier chooses between χ and ψ.

The correctness theorem for GTS$^{\text{RR}}$ is given as follows.

Theorem 5.2. *In a game $\Gamma_{\text{RR}}(M, \varphi, w)$, Heloise the verifier has a winning strategy if φ is true, and Abelard the falsifier has a winning strategy if φ is false.*

Proof. The proof is by induction on φ. Let us see the case for Heloise the verifier with the game state $(\text{Heloise}, \varphi, w)$. The case for Abelard the falsifier is very similar hence skipped.

Let φ be a propositional letter p. If p is true then, by definition, Heloise the verifier has a winning strategy.

Let $\varphi = \neg\psi$. Then, the game continues at $\#w$ for ψ with switched roles, where $v(\#w, \psi) = 0$. Heloise becomes the falsifier at the game position $(\text{Heloise}, \psi, \#w)$. By the induction hypothesis (for Abelard the falsifier), the falsifier has a winning strategy for the play at $\#w$ for ψ. Therefore, Heloise the verifier has a winning strategy at $(\text{Heloise}, \neg\psi, w)$ which forms her winning strategy for φ at $(\text{Heloise}, \varphi, w)$.

The cases for conjunction and disjunction are as expected thus omitted. ∎

Notice that $\Gamma_{\text{RR}}(M, \varphi, w)$ games are not necessarily zero-sum. In order to see this, let

$w \models \neg p$ for a propositional atom p. Stipulate further that $\#(w) = w$. Now, by definition $w \models p$. Therefore, by Theorem 5.2, both Abelard and Heloise have winning strategies in $\Gamma_{\mathrm{RR}}(M, p \wedge \neg p, w)$ where M and w are as above. Therefore, Heloise can have a winning strategy for a false (at the same time true) formula, and likewise for Abelard.

6. Game Semantics for a Connexive Logic

As Wansing puts it, connexive logic is a "comparatively little-known and to some extent neglected branch of non-classical logic" (Wansing, 2015). Even if it is under-studied, its philosophical roots can be traced back to Aristotle and Boethius.

Connexive logic is defined as a system which satisfies the following two schemes of conditionals:

- **Aristotle's Theses**: $\neg(\neg\varphi \to \varphi)$
- **Boethius' Theses**: $(\varphi \to \neg\psi) \to \neg(\varphi \to \psi)$

The philosophical motivation behind the above schemata goes back to early medieval philosophy where unintuitive results about material implication received some interest. In modern days, connexive logics can also be viewed as part of the agenda of relevant logic, which takes the material implication central in its inquiry. In this paper, we will set such historical and philosophical discussions aside and focus on the formal semantics.

As in many different families of non-classical logic, there has been suggested a variety of connexive logics in the literature. In this work, we discuss one of the earliest examples of connexive logics CC, which is due to McCall (McCall, 1966).

The logic CC is axiomatized by adding the scheme $(\varphi \to \varphi) \to \neg(\varphi \to \neg\varphi)$ to the axiomatization of classical propositional logic. The rules of inference for CC is modus ponens and adjunction, which is given as $\vdash \varphi, \vdash \psi \mathrel{.\,\cdot\,.} \vdash \varphi \wedge \psi$. As usual, we consider CC with the standard propositional syntax $\mathcal{L}$.

The semantics for CC is given with 4 truth values: T, t, f and F which can be viewed as "logical necessity", "contingent truth", "contingent falsehood", and "logical impossibility" respectively (Routley & Montgomery, 1968). In CC, the designated truth values are T and t. The truth table for CC is given in Figure 2.

	$\neg$
T	F
t	f
f	t
F	T

$\wedge$	T	t	f	F
T	T	t	f	F
t	t	T	F	f
f	f	F	f	F
F	F	f	F	f

$\vee$	T	t	f	F
T	t	T	t	T
t	T	t	T	t
f	t	T	F	f
F	T	t	f	F

Figure 2. *The truth table for CC.*

Notice that even if there are paraconsistent connexive logics, CC in particular is not paraconsistent. It is possible to obtain paraconsistent connexive logics based on relevant logic RR, which we discussed earlier, or by splitting the satisfaction relation into *negative* and *positive* satisfaction (Wansing, 2015).

The CC truth table, similar to some other non-classical logics considered in this work, exhibits an interesting property that for some binary connectives and truth values, two truth values produce a third truth value. For instance, in CC with a brief abuse of notation, we have $t \wedge f \equiv F$. We generalize this phenomenon as follows.

Definition 6.1. Let L be a n-valued logic where $n \geq 2$, $\{V_i\}_{i \leq n}$ the set of truth-values, and $\{C_j\}_{j \in J}$ be set of binary logical connectives for some index set J. Then, L is said to have the *hereditary condition* if for all $i, i' \leq n, j \in J$, $C_j(V_i, V_{i'})$ evaluates to either

V_i or $V_{i'}$. In short, logical connectives cannot output a truth value different than the input values. This definition can easily be extended to k-ary logical connectives.

Thus, CC lacks the hereditary condition which complicates its game theoretical semantics. For instance, if the verification game is for $t \wedge f$, how can Abelard the falsifier guarantee a win?

First, following our earlier methodology, we introduce 4 players for 4 truth values. The truth value T is forced by Heloise, F by Abelard, t by Aristotle and f by Boethius. We denote the game theoretical semantics for CC with 4 players by GTS$^{\text{CC}}$.

Since CC lacks the hereditary condition, we stipulate that the players form *coalitions*. The idea of forming coalitions in logical games for a win is not new, and its origins in (modal) logic can be found in (Pauly, 2002). In GTS$^{\text{CC}}$, we impose that "Heloise and Aristotle" and "Abelard and Boethius" form two teams, which can be viewed as truth-maker and false-maker coalitions respectively. Now, we define the semantic verification game for CC as follows.

Definition 6.2. The tuple $\Gamma_{\text{CC}} = (\pi, \rho, \sigma, \tau, \delta)$ is a verification game for CC where $\pi = \{$Heloise, Aristole, Boethius, Abelard$\}$, σ is the set of tuples in the form of (Heloise/Aristotle, φ) and (Abelard/Boethius, φ) for well-formed formula φ, τ is composed of singleton game positions as there is no parallel play, and δ is $\{T, t\}$. And, for a game $\Gamma_{\text{CC}}(M, \varphi)$, ρ is given inductively as follows.

- If φ is atomic, the game terminates, and Heloise wins if φ has the truth value T, Aristotle wins if φ has the truth value t, Boethius wins if φ has the truth value f and Abelard wins if φ has the truth value F,
- if $\varphi = \neg\psi$, Heloise assumes Abelard's role, Aristotle assumes Boethius' role, Boethius assumes Aristotle's role and Abelard assumes Heloise's role, and the game continues as $\Gamma_{\text{CC}}(M, \psi)$,
- if $\varphi = \chi \wedge \psi$, false-makers simultaneously choose between χ and ψ,
- if $\varphi = \chi \vee \psi$, truth-makers simultaneously choose between χ and ψ.

In the above definition, game states (Heloise/Aristotle, φ) and (Abelard/Boethius, φ) denote the the truth-maker coalition and false-maker coalition respectively. In short, these coalitions are considered as single agents.

As an alternative to the above definition and a thought experiment, it can be suggested that instead of forming coalitions, players may be allowed to play in parallel just as in GTS$^{\text{LP}}$. In this case, it can be argued that conjunction can be assigned to Abelard and Boethius, and disjunction to Heloise and Aristotle. The following example considers this hypothetical case as well.

Example 6.3. By a slight abuse of notation, let us denote with f, t, F, T the atoms with truth values f, t, F, T respectively. Let us consider the formula $(F \wedge f) \vee (T \wedge t)$ which evaluates to T in CC. In this play, we expect the truth-makers to win. However, they cannot individually achieve a win. If Heloise makes a move to $T \wedge t$, then she needs Aristole to win since Abelard or Boethius can choose whatever they wish at $T \wedge t$. Thus, by forming a coalition, Aristotle and Heloise (the truth-makers) can guarantee a win in this play.

However, following the hypothetical case of allowing parallel plays where individual players play in parallel without a coalition, it can be observed that Heloise cannot guarantee a win. First, she chooses the conjunct $T \wedge t$. But then, Abelard and Boethius can choose whatever they like, in particular t. Thus, Heloise cannot have a winning strategy for a formula with truth value T. This justifies why we need coalitions in GTS$^{\text{CC}}$.

We give the biconditional correctness theorem for GTS$^{\text{CC}}$ as follows.

$(F \wedge f) \vee (T \wedge t)$

Heloise — Aristotle

$F \wedge f$ $T \wedge t$ $F \wedge f$ $T \wedge t$

Abelard Boethius Abelard Boethius Abelard Boethius Abelard Boethius

F f F f T t T t F f F f T t T t

Theorem 6.4. *In a GTS^{CC} verification game $\Gamma_{\mathrm{CC}}(M, \varphi)$,*

- *truth-makers have a winning strategy if and only if φ has the truth value t or T in M,*
- *false-makers have a winning strategy if and only if φ has the truth value f or F in M.*

Proof. We present a proof only for truth-makers as the cases for false-makers are similar. In order to make it easier to follow, we will give proof without being fully formal on the game positions.

[Left-to-Right Direction] Assume that truth-makers have a winning strategy for the verification game $\Gamma_{\mathrm{CC}}(M, \varphi)$, we will show by induction on φ that the truth value of φ is either t or T.

Let φ be a propositional variable p. If truth-makers Heloise and Aristotle have winning strategies, then by definition of GTS$^{\mathrm{CC}}$, the truth value of φ is either T or t.

Now, let $\varphi = \neg\psi$ for some ψ. Assume that truth-makers have a winning strategy for $\Gamma_{\mathrm{CC}}(M, \varphi)$. By the game rules, Heloise and Aristotle assume the role of false-makers, and will have a winning strategy as false makers for the game $\Gamma_{\mathrm{CC}}(M, \psi)$. By the induction hypothesis, then, ψ will be either f or F. Then, by the truth table for CC in Figure 2, φ has the truth value t or T.

As the third step of the induction, let $\varphi = \chi \wedge \psi$ and assume that truth-makers have a winning strategy for $\Gamma_{\mathrm{CC}}(M, \varphi)$. Then, by definition, false-makers make a move and choose either of the conjunct, depending on their strategy. Since truth-makers have a winning strategy for φ whichever conjunct false-makers choose, they will still have a winning strategy. By the induction hypothesis, then the truth-makers have a winning strategy for *both* χ and ψ. By the induction hypothesis, the truth values of χ and ψ are either t or T. The truth table for CC in Figure 2 shows that any conjunction of ts and Ts produces a t or T truth value. Thus, the truth value of $\chi \wedge \psi$ is still t or T which gives us the truth value of φ.

Finally, let $\varphi = \chi \vee \psi$ and assume that truth-makers have a winning strategy for $\Gamma_{\mathrm{CC}}(M, \varphi)$. Then, by definition truth-makers will choose either of the disjunct to employ their winning strategy, say χ without loss of generality. Then, by induction hypothesis, as truth makers also have a winning strategy for χ, the truth value of χ is either t or T. The truth table for CC in Figure 2 shows that any disjunction of a t or T with any other truth value still produces a t or T truth value. Thus, the truth value of $\chi \vee \psi$ is still t or T, independent from the truth value of ψ. Thus, the truth value of φ is either t or T.

[Right-to-left Direction] Let φ have the truth value t or T in M.

If φ is a propositional letter p with a truth value t or T, then truth-makers win the game by definition, hence they possess a winning strategy.

Let $\varphi = \neg\psi$. Then, ψ has the truth value f or F. By the game rules, now the game continues where Heloise and Aristotle are false-makers. By the induction hypothesis (for false-makers), the coalition of Heloise and Astrolabe as false-makers have a winning strategy for ψ. Then, taking one step back, they, as truth-makers, have a winning strategy for φ.

Now, let φ be a conjunction of the form $\chi \wedge \psi$. Since, φ is assumed to have the truth values t or T, according to the truth table for CC in Figure 2, the only way to realize φ as t or T is to have χ and ψ with truth values t or T (or any combination thereof). Thus, whichever move false-makers make at $\chi \wedge \psi$, truth-makers will have a winning strategy, by induction hypothesis. Then, truth-makers will have a winning strategy for ψ.

Finally, let φ be a disjunction of the form $\chi \vee \psi$. Then, truth-makers make a move and choose the disjunct with the truth value t or T. According to the truth table for CC in Figure 2, the only way to realize $\chi \vee \psi$ with a truth value t or T is to have at least one of disjunct with a truth value either t or T. By induction hypothesis, truth-makers have a winning strategy for that disjunct with the truth value t or T. Thus, at $\chi \vee \psi$, truth makers simply choose that disjunct which form the winning strategy of truth makers for φ.

This concludes the proof. ■

It is possible to consider teams and coalitions as individual players with an aim of reducing the verification game to a two-player game. This simply mimics a translation from CC to classical logic where T and t are assigned to T, and F and f are assigned to F.

From a game theoretical perspective, this amounts to loss of information. Identifying two players with different goals to force the play to a "necessary truth" win and to a "contingent truth" win, with a single player to force the play to a "truth" win, does not do justice to the verification game for CC. Players may have similar goals and can form coalitions, yet this by no means entails that they are identical players with identical roles.

The major contribution of CC to GTS is that it introduces *coalitions* into verification games in a natural way. Coalition formation directly reflects the truth table of the logic in question and its philosophical underpinnings. Therefore, some other multi-valued logical systems with similar properties can possibly be given a coalition-based GTS.

7. Game Semantics for Belnap's Four Valued Logic

Belnap's four valued logic (BL, for short) introduces two additional truth values besides the classical ones. The truth value P, as before, represents over-valuation, and N represents under-valuation. Traditionally, P stands for both truth values and N stands for neither of the truth values. As the truth table in Figure 3 indicates, P and N are the fixed-points under negation.

	$\neg$
T	F
P	P
N	N
F	T

$\wedge$	T	P	N	F
T	T	P	N	F
P	P	P	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T	P	N	F
T	T	T	T	T
P	T	P	T	P
N	T	T	N	N
F	T	P	N	F

Figure 3. *The truth table for BL.*

As can be seen from the truth table, similar to CC, BL does not admit the hereditary condition. In the case of BL, with a slight abuse of notation, the problematic formulas are $P \vee N \equiv T$ and $P \wedge N \equiv F$. A Hasse-style truth value lattice for BL is given in Figure 4.

The above Hasse diagram illustrates the standard method to compute disjunction and conjunction of two truth values as the least upper bound and the greatest lower bound

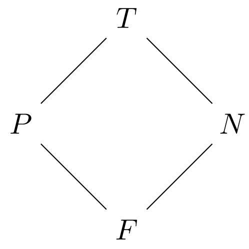


Figure 4. *Hasse Diagram for BL.*

of the two values respectively. Then, with a slight abuse of notation, it is possible to read off $P \vee N \equiv T$ and $P \wedge N \equiv F$ from the diagram. For simplicity, we will take the designated values for BL as $\{P, T\}$.

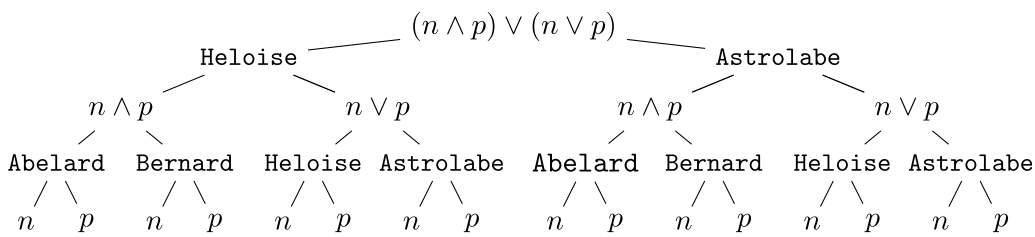
The truth combinations $P \vee N \equiv T$ and $P \wedge N \equiv F$ suggest that our standard approach may not work for BL. For a true formula, Heloise the verifier may have to choose P or N as her end-state for a win. Similarly, Abelard the falsifier may end up with a P or N state as well.

As before, we introduce 4 players for 4 truth values. The truth value T is forced by Heloise, F by Abelard, P by Astrolabe and N by Bernard[1]. In this case, at the beginning of the play, Heloise is the verifier, Abelard is the falsifier, Astrolabe is the paradoxifier, and Bernard is the nullifier. We denote the game theoretical semantics for BL with 4 players by GTS^{BL}.

Similar to earlier games, the first problem is to assign players' roles to the connectives. But this time, BL lacks the hereditary condition. Now, similar to our methodology for GTS^{CC}, we can consider coalitions and form two teams as "Heloise and Astrolabe" and "Abelard and Bernard". In this case, the disjunction will belong to the team of "Heloise and Astrolabe" and the conjunction to 'Abelard and Bernard". Let us test this hypothetical idea in the following example.

Example 7.1. Let us consider the formula $(n \wedge p) \vee (n \vee p)$ which evaluates to T. In this game, we expect the coalition of Heloise the verifier and Astrolabe the falsifier win as a coalition even if neither of the end-nodes has the truth value T. Simply put, in this case Heloise will start playing and pass the ball to Astrolabe for a win.

First, Heloise makes a move and chooses $n \vee p$. Because if she chooses $n \wedge p$, then the coalition of Abelard and Bernard may choose n and win. Then, at $n \vee p$, Astrolabe chooses p which is a win for the coalition of Heloise and Astrolabe.



In the above example and thought experiment, the idea of forming coalitions for BL verification games seem to work. However, from a philosophical perspective, it seems to create a more ambiguous case. For example, for the case when the coalition of Heloise and Astrolabe wins, it is not possible to know the truth value of the formula in question. Is it T (after Heloise) or P (after Astrolabe)? In the case of GTS^{CC}, the aforementioned

[1]After Abelard's rival Bernard of Clairvaux.

ambiguity might have been considered relatively less serious as the resulting truth value of the coalition-led game was either one of the trues or one of the falses. Simply put, the goal and the purpose of the coalitions in GTS$^{\mathrm{CC}}$ was clear: truth-makers and false-makers formed their respective teams. However, it does not seem to be the case in GTS$^{\mathrm{BL}}$. In order to remedy this problem, we suggest a radical version of simultaneous play for GTS$^{\mathrm{BL}}$. We associate conjunction with Abelard the falsifier, Astrolabe the paradoxifier and Bernard the nullifier; and disjunction with Heloise the verifier, Astrolabe the paradoxifier and Bernard the nullifier. Also, we take special care of the problematic formulas.

We define the semantic verification game for BL for 4 players as follows in a rather unusual way. Notice that for simplicity in the argument we associate players with their roles.

Definition 7.2. The tuple $\Gamma_{\mathrm{BL}} = (\pi, \rho, \sigma, \tau, \delta)$ is a verification game for BL where $\pi = \{\text{Heloise, Astrolabe, Abelard, Bernard}\}$, σ is the set of tuples (p_i, φ) for $p_i \in \pi$ and well-formed formula φ, and δ is $\{T, P\}$. For a game $\Gamma_{\mathrm{BL}}(M, \varphi)$, τ is given inductively for the positions (p_i, φ) in σ as follows,

- if $\varphi = \neg\psi$, then,
  - if $\{(\text{Abelard}, \neg\psi)\} \in \tau$ then, $\{(\text{Heloise}, \psi)\} \in \tau$,
  - if $\{(\text{Heloise}, \neg\psi)\} \in \tau$ then, $\{(\text{Abelard}, \psi)\} \in \tau$,
  - if $\{(\text{Astrolabe}, \neg\psi)\} \in \tau$ then, $\{(\text{Astrolabe}, \psi)\} \in \tau$,
  - if $\{(\text{Bernard}, \neg\psi)\} \in \tau$ then, $\{(\text{Bernard}, \psi)\} \in \tau$,
- if $\varphi = \chi \wedge \psi$, then $\{(\text{Abelard}, \chi \wedge \psi), (\text{Astrolabe}, \chi \wedge \psi), (\text{Bernard}, \chi \wedge \psi)\} \in \tau$,
- if $\varphi = \chi \vee \psi$, then $\{(\text{Heloise}, \chi \vee \psi), (\text{Astrolabe}, \chi \vee \psi), (\text{Bernard}, \chi \wedge \psi)\} \in \tau$;

and, finally, ρ is given inductively as follows

- If φ is atomic, the game terminates, and Heloise wins if φ has the truth value T, Astrolabe wins if φ has the truth value P, Bernard wins if φ has the truth value N and Abelard wins if φ has the truth value F,
- if $\varphi = \neg\psi$, Heloise assumes Abelard's role, Abelard assumes Heloise's role, Astrolabe and Bernard keep their previous roles, and the game continues as $\Gamma_{\mathrm{BL}}(M, \psi)$,
- if $\varphi = \chi \wedge \psi$ where only Bernard has a winning strategy for $\Gamma_{\mathrm{BL}}(M, \chi)$ and only Astrolabe has a winning strategy for $\Gamma_{\mathrm{BL}}(M, \psi)$, then Abelard wins,
- if $\varphi = \chi \wedge \psi$, for other cases, Abelard, Astrolabe and Bernard choose simultaneously between χ and ψ,
- if $\varphi = \chi \vee \psi$ where only Bernard has a winning strategy for $\Gamma_{\mathrm{BL}}(M, \chi)$ and only Astrolabe has a winning strategy for $\Gamma_{\mathrm{BL}}(M, \psi)$, then Heloise wins,
- if $\varphi = \chi \vee \psi$, for other cases, Heloise, Astrolabe and Bernard choose simultaneously between χ and ψ.

The unusual game rules about $P \wedge N$ and $P \vee N$ stem from the truth table for BL, and the price we have to pay is to incorporate the existence of winning strategies into the semantics - similar to what we observed for LP when we inquired about the possibility of obtaining a biconditional correctness theorem.

It is important to note that this unusual semantics underlines the significance of game composition in non-classical games. In other words, if the winning strategy of Bernard and the winning strategy of Astrolabe is composed disjunctively, what we obtain is a win for Heloise. Similarly, if the winning strategy of Bernard and the winning strategy of Astrolabe is composed conjunctively, what we obtain is a win for Heloise.

The one-directional correctness theorem for GTS$^{\mathrm{BL}}$ is given as follows.

Theorem 7.3. *In a GTS^{BL} verification game* $\Gamma_{\mathrm{BL}}(M, \varphi)$,

- *Heloise the verifier has a winning strategy if φ evaluates to T in M,*
- *Abelard the falsifier has a winning strategy if φ evaluates to F in M,*
- *Astrolabe the paradoxifier has a winning strategy if φ evaluates to P in M,*
- *Bernard the nullifier has a winning strategy if φ evaluates to N in M.*

Proof. The proof is an induction on the complexity of φ for each players respecting the different cases for $N \wedge P$ and $N \vee P$, and is very similar to the previous cases. Hence, few cases for Heloise and Bernard will be given in this proof.

Let us start with Heloise the verifier at position (Heloise, φ). The cases for propositional variables and negation is immediate. The case for disjunction $N \vee P$ follows directly from the definition and is a win for Heloise.

For the cases for other disjunctions, Heloise the verifier chooses the disjunct with the truth value T which gives her a win by the induction hypothesis. And this win constitutes her winning strategy for the original formula φ.

For conjunction $\varphi = \chi \wedge \psi$, by the truth table for BL in Figure 3, there is only one option. That is, both ψ and χ are true. Thus, whatever Abelard the falsifier, Bernard the nullifier or Astrolabe the paradoxifier chooses, their choice will have the truth value T. By the induction hypothesis, then Heloise the verifier has a winning strategy for whatever choice the other players make for the conjuncts. And that is the winning strategy for Heloise for φ.

For Bernard the nullifier at the game position (Bernard, φ), the cases for propositional variables and the negation follow directly from the definitions.

For $\varphi = \chi \wedge \psi$ with a truth value N, it can be observed in the truth table for BL in Figure 3 that at least one of the conjuncts must have the truth value N, say χ without loss of generality. Then, Bernard the nullifier chooses χ. By the induction hypothesis, Bernard the nullifier has a winning strategy for the verification game for χ at (Bernard, χ). Therefore, choosing χ, the conjunct with a truth value N is Bernard the nullifier's winning strategy at (Bernard, φ).

If $\varphi = \chi \wedge \psi$, and χ and ψ have truth values N and P respectively, the theorem holds vacuously for Bernard the nullifier as he is not allowed to make a move at this point by the game rules.

For $\varphi = \chi \vee \psi$ with a truth value N, then by the truth table for BL in Figure 3, at least one of the disjuncts has the truth value N, and Bernard the nullifier chooses that conjunct, say χ. By the induction hypothesis, he has a winning strategy at the game state (Bernard, χ). Then, he has a winning strategy for φ at state (Bernard, φ).

Similarly, if $\varphi = \chi \vee \psi$, and χ and ψ have truth values N and P respectively, the theorem holds vacuously for Bernard the nullifier as he is not allowed to make a move at this point by the game rules.

As we argued, the cases for Abelard the falsifier and Astrolabe the paradoxifier are very similar, hence skipped.

This concludes the proof. ∎

The converse of the theorem does not immediately hold for obvious reasons. For example, in the game $\Gamma_{\mathrm{BL}}(M, p \vee n)$, Astrolabe the paradoxifier and Bernard the nullifier have winning strategies as well, but this does not entail that the formula has the truth value P or N.

Theorem 7.3 starts from the truth values and establishes the existence of (non-exclusive) winning strategies. Parallel plays introduce the possibility of the existence of winning strategies for multiple players. Therefore, it is important to ask whether the converse of Theorem 7.3 holds or more importantly, *under which conditions* the converse of Theorem 7.3 holds. The following result suggests a restriction where the existence of winning strategies necessarily determine the truth value of the formula in question, similar to Theorem 3.4. It also reflects the algebraic structure of BL given in Figure 4.

Theorem 7.4. *In a GTS^{BL} verification game $\Gamma_{\text{BL}}(M, \varphi)$,*

- *If Heloise the verifier has a winning strategy, then φ evaluates to T,*
- *If Abelard the falsifier has a winning strategy, then φ evaluates to F,*
- *If only Astrolabe the paradoxifier has a winning strategy, then φ evaluates to P,*
- *If only Bernard the nullifier has a winning strategy, then φ evaluates to N*

in the model M.

Proof. The proof is by induction on the complexity of φ for each player.

The cases for Heloise the verifier and Abelard the falsifier are similar to previous cases, except the cases for $\chi \wedge \psi$ and $\chi \vee \psi$ where χ, ψ are formulas with truth values N and P.

Let us consider the case for $\varphi = \chi \wedge \psi$ where only Bernard has a winning strategy for $\Gamma_{\text{BL}}(M, \chi)$ and only Astrolabe has a winning strategy for $\Gamma_{\text{BL}}(M, \psi)$. Then, by the relevant parts of this theorem, χ evaluates to N and ψ evaluates to B. Then by the truth table for BL given in Figure 3 and by Theorem 7.3, φ evaluates to T and Heloise admits a winning strategy.

The other cases including $\chi \vee \psi$ where χ, ψ are formulas with truth values N and P respectively are very similar.

Next is the case for Astrolabe the paradoxifier. The cases for propositional variables and the negation is immediate as Astrolabe keeps his role in the case of negation.

Let us assume $\varphi = \chi \wedge \psi$ where *only* Astrolabe has a winning strategy in $\Gamma_{\text{BL}}(M, \varphi)$. By this restriction, the case where only Bernard has a winning strategy for $\Gamma_{\text{BL}}(M, \chi)$ and only Astrolabe has a winning strategy for $\Gamma_{\text{BL}}(M, \psi)$ is excluded (as also it was covered earlier).

Now, for $\varphi = \chi \wedge \psi$, Abelard, Astrolabe and Bernard choose simultaneously. Since, only Astrolabe has a winning strategy, neither Abelard nor Bernard has winning strategies for neither of the conjuncts χ nor ψ. Astrolabe now chooses one of the conjuncts as part of his winning strategy. Assume he chooses χ without loss of generalization. By the induction hypothesis, χ evaluates to the truth value P. By the truth table given for BL in Figure 3, we have two cases for the truth value of φ: P or F.

Next, we eliminate the cases for F. Formula φ evaluates to F only if ψ is F or N. By the relevant part of this theorem and by the assumption that Abelard cannot have a winning strategy for for φ, it follows that ψ cannot be F. Similarly, ψ cannot be N. If it was, then by Theorem 7.3 Bernard the nullifier would have a winning strategy for $\Gamma_{\text{BL}}(M, \psi)$. Since, by the game rules, Bernard is allowed to make a move for $\varphi = \chi \wedge \psi$, then he would have a winning strategy for $\Gamma_{\text{BL}}(M, \varphi)$ which would contradict with the assumption that only Astrolabe had a winning strategy in that game. Thus, φ can only have the truth value P.

The case for disjunction for Astrolabe is similar, hence omitted.

Moreover, the case for Bernard the nullifier is very similar to that of Astrolabe's, hence left to the reader.

This completes the proof. ∎

Theorem 7.4 shows how restrictions on the existence of winning strategies guarantee the truth value of the formula in question.

Logics FDE and BL seem rather similar. There can be given effective translations between their truth values, and their philosophical motivations look closely related. However, the games we introduced for them are very different. The logical and technical differences between BL and FDE surface more clearly from the view point of GTS.

Furthermore, Theorem 7.4 suggests how we should understand the dominance of strategies in $\mathrm{GTS}^{\mathrm{BL}}$. The theorem states how the existence of winning strategies - without depending on the existence of winning strategies for other players - determine (or do not determine) the truth value of the formula in question. This is a fruitful and interesting future research direction and relates to the elimination of dominated strategies as a solution concept in game theory. We leave such extension to future work.

8. Translating Games

In a recent work, an efficient translation between three-valued logics and modal logic S5 was given (Kooi & Tamminga, 2013). A model of modal logic S5 is defined as a tuple (W, R, V) where W is a non-empty set, R is an equivalence relation on $W \times W$ and V is the standard valuation function. The logic S5 generates equivalence classes and the modal operator can be used to define a new paraconsistent negation $\neg\Box$ (Béziau, 2005). Furthermore, there have been suggested various consequence relations between modal S4 and various intuitionistic and dual-intuitionistic logics (Shramko, 2016). For that reason, it can be inquired whether there exist further connections between modal and paraconsistent logics. In what follows, we present a game theoretical connection between modal and paraconsistent logic based on the translations in (Kooi & Tamminga, 2013; Shramko, 2016).

Briefly, the language $\mathcal{L}_M$ of S5 is obtained by taking the syntactic closure of propositional logic with the possibility modal operator $\Diamond$ and the necessity modal operator $\Box$ in the usual way. Semantically, for $w \in W$ in a model M, we have the following.

$$\begin{array}{lll} M, w \models \Box\varphi & \text{iff} & \forall v. Rwv \to M, v \models \varphi \\ M, w \models \Diamond\varphi & \text{iff} & \exists v. Rwv \land M, v \models \varphi \end{array}$$

Furthermore, in S5, a formula $\Box\varphi$ is true at a state w in a model M if and only if it is true at *all* $v \in [w]$ where $[w]$ denotes the partition of W that contains w. Dually, a formula $\Diamond\varphi$ is true at state w in a model M if and only if it is true at *some* $v \in [w]$.

GTS for the classical modal logic is well-known. “Diamond” formulas are assigned to Heloise the verifier whereas the “Box” formulas are assigned to Abelard the falsifier. Also, similar to the RR, formulas in $\mathcal{L}_M$ are associated with a possible world. For modal formulas, the position of the game token for the next move is determined by the accessibility relation.

In this section, we give a translation of LP (and K3) into S5 via GTS. The translation is built on the following observation:

> “In an S5-model there are three mutually exclusive and jointly exhaustive possibilities for each atomic formula p: either p is true in all possible worlds, or p is true in some possible worlds and false in others, or p is false in all possible worlds” (Kooi & Tamminga, 2013).

The translations $\mathsf{Tr}_{\mathrm{LP}} : \mathcal{L} \mapsto \mathcal{L}_M$ and $\mathsf{Tr}_{\mathrm{K3}} : \mathcal{L} \mapsto \mathcal{L}_M$ for LP and K3 respectively are given as follows, where p is a propositional variable (Kooi & Tamminga, 2013).

$$\begin{array}{rlrl} \mathsf{Tr}_{\mathrm{LP}}(p) &= \Diamond p & \mathsf{Tr}_{\mathrm{LP}}(\varphi \land \psi) &= \mathsf{Tr}_{\mathrm{LP}}(\varphi) \land \mathsf{Tr}_{\mathrm{LP}}(\psi) \\ \mathsf{Tr}_{\mathrm{K3}}(p) &= \Box p & \mathsf{Tr}_{\mathrm{K3}}(\varphi \land \psi) &= \mathsf{Tr}_{\mathrm{K3}}(\varphi) \land \mathsf{Tr}_{\mathrm{K3}}(\psi) \\ \mathsf{Tr}_{\mathrm{LP}}(\neg\varphi) &= \neg\mathsf{Tr}_{\mathrm{K3}}(\varphi) & \mathsf{Tr}_{\mathrm{LP}}(\varphi \lor \psi) &= \mathsf{Tr}_{\mathrm{LP}}(\varphi) \lor \mathsf{Tr}_{\mathrm{LP}}(\psi) \\ \mathsf{Tr}_{\mathrm{K3}}(\neg\varphi) &= \neg\mathsf{Tr}_{\mathrm{LP}}(\varphi) & \mathsf{Tr}_{\mathrm{K3}}(\varphi \lor \psi) &= \mathsf{Tr}_{\mathrm{K3}}(\varphi) \lor \mathsf{Tr}_{\mathrm{K3}}(\psi) \end{array}$$

The translation is a co-induction, and generates fully modalized formulas. For example, the translation of the LP formula $p \land \neg q$ is $\Diamond p \land \neg\Box q$ in S5 - every proposition generates a modal formula in S5. More interestingly, the translation of the LP formula $p \wedge \neg p$ is $\Diamond p \wedge \neg\Box p$ in S5, which is equivalent to $\Diamond p \wedge \Diamond\neg p$ not necessarily inconsistent in S5. However, the same formula $p \wedge \neg p$ translates from K3 to S5 as $\Box p \wedge \Box\neg p$ which is contradictory. Moreover, $\mathsf{Tr}_{\mathrm{LP}}(p \vee \neg p) = \Diamond p \vee \Diamond\neg p$ which holds in S5, while $\mathsf{Tr}_{\mathrm{K3}}(p \vee \neg p) = \Box p \vee \Box\neg p$ which does not hold in general in S5.

As the authors underlined, for fully modalized formulas in S5, a formula is true somewhere in an S5 model if and only if it is true everywhere in the model. This fact is due to the frame properties of S5. We refer the reader to (Kooi & Tamminga, 2013) for a detailed exposition of the translation and its correctness proof.

Given $\Gamma_{\mathrm{LP}} = (\pi, \rho, \sigma, \tau, \delta)$, we define $\Gamma_{\mathrm{S5}} = (\pi', \rho', \sigma', \tau', \delta')$ as follows: $\pi' = \{\text{Heloise}, \text{Abelard}\}$, $\delta' = \{1\}$; ρ', σ', and τ' are rules, positions and position of game token of verifications games for (classical) S5, respectively. Similarly, a win for the verifier is the situation when the play terminates with a true atom. Dually, a player terminated with a false atom brings a win to the falsifier.

Furthermore, given an LP model M with valuation V, there exists a corresponding S5 model M' (Kooi & Tamminga, 2013). Let us describe how $M' = (W', V')$ is constructed. Let $W' = \{w, w'\}$, and for all propositional variables, we define V' as follows for truth values T, F and P of LP:

$$\begin{array}{lll} V'(p) = W' & \text{iff} & V(p) = \{T\} \\ V'(p) = \{w\} & \text{iff} & V(p) = \{P\} \\ V'(p) = \emptyset & \text{iff} & V(p) = \{F\} \end{array}$$

That is, true formulas are true everywhere in the S5 model, false formulas are false everywhere, paradoxical formulas are true somewhere - not everywhere, not nowhere.

The following example illustrates the games we have defined.

Example 8.1. Let us reconsider Example 3.1. In this example, the formula $(p \wedge t) \vee (p \wedge f)$ which evaluates to P in LP was examined. In this LP game, Astrolabe the paradoxifier has a winning strategy.

Now, $\mathsf{Tr}((p \wedge t) \vee (p \wedge f)) = (\Diamond p \wedge \Diamond t) \vee (\Diamond p \wedge \Diamond f)$. In this game in S5, we expect both players have a winning strategy.

The first move is due to Heloise. She chooses $\Diamond p \wedge \Diamond t$. Otherwise, if she chooses $\Diamond p \wedge \Diamond f$, then Abelard may choose $\Diamond f$ which would result in Heloise not having a winning strategy.

Now, at $\Diamond p \wedge \Diamond t$, Abelard chooses $\Diamond p$. Otherwise, he cannot possibly win. At $\Diamond p$, however, he chooses the state in S5 model (that is w according to the model defined earlier) at which the atom p is false. But, by the observation that $V(\neg p) = V(p)$, the very same state in M' also verifies p. So, this move brings both players a win as $\neg p$ has the same extension as p.

Thus, both players have a winning strategies in the S5 game Γ_{S5} for the given formula in this model.

The following theorem establishes the correctness of the translation for LP.[1]

Theorem 8.2. *Let $\Gamma_{\mathrm{LP}}(M, \varphi)$ be given. Then, there exists a corresponding S5 model M' and,*

- *if Heloise the verifier has a winning strategy in $\Gamma_{\mathrm{LP}}(M, \varphi)$, then she has a winning strategy in $\Gamma_{\mathrm{S5}}(M', \mathsf{Tr}_{\mathrm{LP}}(\varphi))$,*
- *if Abelard the falsifier has a winning strategy in $\Gamma_{\mathrm{LP}}(M, \varphi)$, then he has a winning strategy in $\Gamma_{\mathrm{S5}}(M', \mathsf{Tr}_{\mathrm{LP}}(\varphi))$,*
- *if only Astrolabe the paradoxifier has a winning strategy in $\Gamma_{\mathrm{LP}}(M, \varphi)$, then both*

[1] A similar theorem can also be given for K3 very easily.

Abelard and Heloise have winning strategies in $\Gamma_{\mathrm{S5}}(M', \mathsf{Tr}_{\mathrm{LP}}(\varphi))$.

Proof. The theorem is given for LP and S5 and is a co-induction on φ. For this reason, we assume the correctness of a similar theorem for K3 and S5 whose proof requires the correctness of the current theorem. Here, we present only the proof of the translation between LP and S5.

First of all, let us observe again how $M' = (W', V')$ is constructed. Let $W' = \{w, w'\}$, and for propositional variable p, we define V' as follows for truth values T, F and P of LP:

$$\begin{array}{lll} V'(p) = W' & \text{iff} & V(p) = \{T\} \\ V'(p) = \{w\} & \text{iff} & V(p) = \{P\} \\ V'(p) = \emptyset & \text{iff} & V(p) = \{F\} \end{array}$$

Now, assume that Heloise the verifier has a winning strategy for φ in LP. Let us proceed by induction on φ. If φ is a propositional letter p, then p is true in LP. Then, it translates to S5 as $\Diamond p$, which is a turn for Heloise the verifier. Then, the game in S5 starts by Heloise with $\Diamond p$, and she can choose any state in the model as p is true everywhere. This is her winning strategy in $\Gamma_{\mathrm{S5}}(M', \mathsf{Tr}_{\mathrm{LP}}(\varphi))$.

For $\varphi = \neg\psi$, suppose Heloise the verifier has a winning strategy for $\neg\psi$ in LP. By the translation, she has a winning strategy for $\neg\mathsf{Tr}_{\mathrm{K3}}(\psi)$ in S5. So, by the assumed similar theorem for K3 and S5, the falsifier has a winning strategy in S5 for $\mathsf{Tr}_{\mathrm{K3}}(\psi)$. Then, in S5 Heloise the verifier has a winning strategy for $\neg\mathsf{Tr}_{\mathrm{K3}}(\psi)$ which is $\mathsf{Tr}_{\mathrm{LP}}(\neg\psi)$. Thus, Heloise the verifier has a winning strategy for $\mathsf{Tr}_{\mathrm{LP}}(\varphi)$ in S5.

The cases for conjunction and disjunction are immediate. Also, the case for Abelard the falsifier is very similar, hence skipped.

The case for Astrolabe the paradoxifier is interesting. Assume that only Astrolabe the paradoxifier has a winning strategy for φ in LP. Then by Theorem 3.4, φ has the truth value P.

As the first step of the induction, assume $\varphi = p$ for a propositional variable p. This translates as $\Diamond p$ and is true at w in M', as observed already. However, as $V(p) = V(\neg p)$ in LP, $\Diamond\neg p$ (which is $\mathsf{Tr}_{\mathrm{LP}}(\neg p)$) is also true at w. Heloise the verifier makes a move to w, which her only possibility to verify $\Diamond p$. Yet, this move also constitutes a win for Abelard the falsifier. Thus, both players have winning strategies for $\varphi = p$ in $\Gamma_{\mathrm{S5}}(M', \mathsf{Tr}_{\mathrm{LP}}(\varphi))$.

The case for negation for Abelard the paradoxifier is obvious as the negation of a paradoxical formula is paradoxical. Similarly, the cases for the binary connectives are immediate, hence skipped. ■

Next, we discuss the converse of Theorem 8.2. For this purpose, we start with the following definition.

For an LP valuation v, and a model M of S5, v and M are said to be $\mathsf{Tr}_{\mathrm{LP}}$-*equivalent* if for all $\varphi \in \mathcal{L}$ we have (i) $1 \in v^*(\varphi) \Leftrightarrow M \models_{S5} \mathsf{Tr}_{\mathrm{LP}}(\varphi)$, and (ii) $0 \in v^*(\varphi) \Leftrightarrow M \not\models_{S5} \mathsf{Tr}_{\mathrm{K3}}(\varphi)$, where v^* is the (truth table) function based on v that maps *formulas* to truth values of LP. Based on various results in (Kooi & Tamminga, 2013), we now prove the following.

Theorem 8.3. *Let M be an S5 model, $\varphi \in \mathcal{L}$ with an associated verification game $\Gamma_{\mathrm{S5}}(M, \varphi)$. Then, there exists an LP model M' and a game $\Gamma_{\mathrm{LP}}(M', \varphi)$ where,*

- *if Heloise the verifier has a winning strategy for $\Gamma_{\mathrm{S5}}(M, \varphi)$ at each point in M, then Heloise has a winning strategy in $\Gamma_{\mathrm{LP}}(M', \varphi)$,*
- *if Abelard the falsifier has a winning strategy for $\Gamma_{\mathrm{S5}}(M, \varphi)$ at each point in M, then Abelard has a winning strategy in $\Gamma_{\mathrm{LP}}(M', \varphi)$,*
- *if Heloise or Abelard has a winning strategy for $\Gamma_{\mathrm{S5}}(M, \varphi)$ at some points but not all in M, then Astrolabe has a winning strategy in $\Gamma_{\mathrm{LP}}(M', \varphi)$.*

Proof. As we observed, the propositions that are true *everywhere* are associated with the LP truth value T, the propositions that are true *nowhere* with F, and the propositions that are true *somewhere* with P. In (Kooi & Tamminga, 2013), it was shown that these LP and S5 models are $\mathsf{Tr}_{\mathrm{LP}}$-equivalent.

Based on this observation, if Heloise has a winning strategy for $\Gamma_{\mathrm{S5}}(M, \varphi)$ at all points in M, it means that, by the aforementioned translation, φ has a truth value T in LP. Then by Theorem 3.3, Heloise has a winning strategy in $\Gamma_{\mathrm{LP}}(M', \varphi)$. Similarly, if Abelard has a winning strategy for $\Gamma_{\mathrm{S5}}(M, \varphi)$ at all points in M, it means that, by the aforementioned translation, φ has a truth value F in LP. Then by Theorem 3.3, Abelard has a winning strategy in $\Gamma_{\mathrm{LP}}(M', \varphi)$. Finally, if Astrolabe has a winning strategy for $\Gamma_{\mathrm{S5}}(M, \varphi)$ at some points in M, it means that, by the aforementioned translation, φ has a truth value P in LP. Then by Theorem 3.3, Astrolabe has a winning strategy in $\Gamma_{\mathrm{LP}}(M', \varphi)$. ■

The translation between LP and S5 via game semantics brings along the question of apply the same methodology to a (possible) translation between BL and a modal logic. We leave this to future work.

9. *Difficult* Logics

There are some well-known and well-studied paraconsistent logics in the literature which have not been discussed in this work. Let us now explain why the well-known Brazilian and Canadian schools of paraconsistency do not seem to fit in our Hintikkan approach.

The da Costa system is one of the most well-known and well-studied systems of paraconsistent logic (da Costa, 1974; da Costa et al., 2007). In order to control inconsistencies, da Costa systems introduce a weaker negation alongside a consistency operator $\circ$. This operator brings the meta-logical condition of consistency down to the object level. In this syntax, $\circ\varphi$ reads that the formula φ is consistent. Moreover, the classical negation can also be reclaimed in the da Costa systems.

From a game theoretical perspective, the rather *ad hoc* consistency operator seems unusual as it does not seem to have an immediate game theoretical counterpart, to the best of our knowledge. There have been attempts to give Kripkean possible world semantics and topological semantics for da Costa logics, yet such works simply take this operator as a given function (Baaz, 1986; Başkent, 2016). Similarly, the Logic of Formal Inconsistency (LFI, for short) extends da Costa systems and seems to suffer from similar game theoretical issues (Carnielli, Coniglio, & Marcos, 2007).

Game theoretical problems for da Costa logics are rooted in the fact that players in our approach cannot test the consistency of the formulas when they are making moves. This would require an (omniscient and omnipotent) auxiliary player whose task is to check the consistency of the formulas. Since such a player cannot be associated with truth values, we chose not to implement this idea in this work. Moreover, the game theoretical connection between the consistency of a formula and the winning strategies is not clear, as the logics we have discussed have demonstrated. Therefore, admitting a consistency-forcer player seems difficult, if not impossible in our Hintikkan approach.

The Canadian school of paraconsistency, which is widely known as *Preservationism* and pioneered by Jennings and Schotch, also falls outside the scope of the current paper. In Jennings and Schotch's work, modal truth conditions are extended from a binary accessibility relation to an n-ary accessibility relation R. In this way, their system *preserves* the rule of necessitation and the monotonicity principle of modal logic (Schotch, Brown, & Jennings, 2009). The semantics for the necessity modality is given as follows for a possible world w.

$$w \models \Box\varphi \quad \text{if and only if} \quad \forall v_1, \ldots, v_n(Rwv_1 \ldots v_n \to \exists i \leq n.v_i \models \varphi)$$

In this case it is not difficult to see how some possible worlds can satisfy the modal contradictions in the form of $\Box(\varphi \wedge \neg\varphi)$. Additionally, one of the technical strengths of the preservationist account is its interdependence with various algebraic and lattice theoretical notions.

The above definition introduces an additional layer of complexity for game theoretical semantics for preservationist logics. In the classical, binary modal case, the accessibility relation dictates how the game proceeds in a deterministic fashion. For boxed-formulas, Abelard the falsifier chooses whichever accessible state he wishes, and for diamond-formulas Heloise the verifier chooses one of the accessible states. However, in preservationism, for boxed-formulas Abelard the falsifier must first choose a string of accessible states (that is a vector of the form $v_1 \ldots v_n$) among many possibilities. This choice implicitly includes another choice, which is represented by the existential quantifier in the consequent to determine the precise point in the vector which satisfies the formula. This would amount to two choices disguised in one move. Yet another complication is that such a move involves formula dependency, and the game rules must reflect this fact. For this reason, we have not considered preservationist paraconsistent systems in this work, even if they seem to provide a rich mathematical machinery and interesting applications, especially for IF-logic or dependency logic (Mann et al., 2011; Väänänen, 2007).

10. Conclusion

In this work, we discussed how different non-classical logics require different verification games. We observed that some logics require additional players, some require concurrent play and some require coalitions. Most importantly, paraconsistent logics break the immediate connection between truth values and winning strategies. In addition to that, in some games, the knowledge of the existence of winning strategies for the opponents may be required to determine the truth value of the formula. This directly relates the discussion to epistemic game theory in a way that was not addressed by IF-logic. Furthermore, we observed that for some logics, admitting winning strategies do not conclusively say anything about the truth value of the formula in question.

Additionally, the current work can be seen as an attempt to broaden the semantical basis of non-classical logic. A vast majority of work on the subject presents an axiomatic, proof-theoretical approach, and the semantical analysis of the subject is usually found trapped between truth-tables and algebraic semantics. Game theoretical semantics, on the other hand, has the potential to position itself within the intersection of both research programs. It introduces non-classical logics more systematically to the Hintikkan agenda of game theoretical semantics, and at the same time, encourages further applications of the Hintikkan approach in non-classical logical domains - which, for various philosophical reasons, remained not-so-influential for non-classical logicians.

Game theory is the study of rational, interactive decision making. In this work, we studied the interactive decisions in logic and games, and did not discuss the rationality of the players, apart from their hidden assumption to choose the best moves to win the play of the game. Potential future work in this area is to relate non-classical logical games to (ir-)rationality. In this framework, it is meaningful to ask how players' rationality changes from one logic to the another, and how the rationality of players can be defined for inconsistent logics and their verification games.

Our choice of logics can be seen as limited. This is most certainly true. Within the reasonable limits of a research article, we considered few propositional logics, one from each family, and discussed their game theoretical semantics. Each family of non-classical logics contains a wide variety of logical systems with some modal and first-order extensions. For practical reasons, we limited ourselves to well-known, relatively well-studied propositional non-classical logics. This should serve as an indication to show the breadth and depth of this research project and its future potential. Therefore, we leave the extension of this approach to modal and first-order non-classical logics to future work.

Our approach here can be seen as a case for the plurality of logic (Beall & Restall, 2006). The well-known classical GTS is essentially a very narrow, limited case with many additional and auxiliary game theoretical and logical assumptions and restrictions. Once those assumptions are set aside (or at least questioned) for various reasons, GTS turns out to be expressive enough for a variety of non-classical logics as we have exemplified.

Acknowledgements

First of all, I am grateful for the skeptical feedback of the referees of this journal, which helped me improve the paper. I acknowledge the help of Marco Panza for his careful reading of the earlier version of this paper, of Gabriel Sandu for hosting me in Helsinki for a short while in 2013, and of Guy McCusker for providing valuable insight for the future of this project. I acknowledge the useful feedback of Diderik Batens, Marco Panza, Ivano Ciardelli and Jeroen Groenendijk. Earlier versions of this paper was presented in Paris, Nancy, Ghent, Vienna, London, Stockholm, Poznań, Kolkata, Birmingham and Taipei in various events. The paper is an extended version of (Başkent, 2015) with full proofs and many additional results extending it to a broader class of logical systems.

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