Associated Primes for Cohomology Modules

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1. Introduction

Let k be a field of finite characteristic p, and G a finite group acting on the left on a finite dimensional k-vector space V. Then the dual vector space V^* is naturally a right kG-module, and the symmetric algebra of the dual, $R := \mathrm{Sym}(V^*)$, is a polynomial ring over k on which G acts naturally by graded algebra automorphisms, and if k is algebraically closed can be regarded as the space $k[V]$ of polynomial functions on V. The G-fixed points of R under this action form a ring, which we denote by R^G and call the ring of invariants. If k is algebraically closed, R^G can be regarded as the set of G-invariant polynomial functions on V, or the ring of coordinate functions on the quotient space V/G. The ring of invariants R^G is the central object of study in invariant theory. The situation becomes modular when we assume p divides the order of G. Let P be a (fixed) Sylow-p-subgroup of G.

Since the ring of invariants R^G coincides with the zeroth cohomology $H^0(G,R)$, we can regard R^G as the zeroth degree part of the cohomology ring $H^*(G,R)$, and as such, the higher cohomology modules $H^i(G,R)$ become R^G-modules via the cup product. One can often learn more about the structure of modular rings of invariants by studying these higher cohomology modules; for example, in [3] Ellingsrud and Skjelbred showed that $H^i(G,R)$ is Cohen-Macaulay for G cyclic of order p. They then used this result to find a formula for the depth the ring of invariants R^G in this case. This approach was also used in [7], [9] and [10] to answer questions about the depth or Cohen-Macaulay property of modular invariant rings.

If $X < G$, we may define a mapping $\mathrm{Tr}^G_X : R^X \to R^G$ as follows: let S be a set of right coset representatives of X in G. Then we define

$$\mathrm{Tr}^G_X(x) := \sum_{g \in S} xg. \tag{1}$$

This mapping is often called the *relative transfer*, and induces mappings $\mathrm{Tr}^G_X : H^i(X,R) \to H^i(G,R)$ also called the relative transfer. Both are surjective when the index of X in G is coprime to p. The image of the transfer map $\mathrm{Tr}^G_X(R^X)$ is an ideal in R^G called the *relative transfer ideal* which we denote by I^G_X. We may generalise this definition and define

$$I^G_\chi := \sum_{X\in\chi} I^G_X$$

for any set χ of subgroups of G. Relative transfer ideals and their radicals have been studied widely in connection with modular invariant theory. For example it is known that the quotient ring $R^G/\sqrt{I^G_{<P}}$[1] is always Cohen-Macaulay (see [6]).

Let $H^+(G,R)$ denote the set of positive degree elements of the cohomology ring $H^*(G,R)$, that is, we define $H^+(G,R) := \oplus_{i>0} H^i(G,R)$. The main purpose of this paper is to prove the following:

Theorem 1.1. *Let $\mathfrak{p}$ be an associated prime ideal of the R^G-module $H^+(G,R)$. Then $\mathfrak{p} = \sqrt{I^G_\chi}$ for some set χ of subgroups of G.*

Remark: The relative transfer ideals I^G_χ defined above were first studied by Fleischmann ([5]), who proved the following formulae:

$$\sqrt{I^G_\chi} = (\bigcap_{X\in\chi'} ((g-1)V^* |\ g\in X)R) \cap R^G = (\bigcap_{X\in\chi'} \mathcal{I}(V^X)) \cap R^G \qquad (2)$$

where $\chi' := \{Q \le P|\ Q \not\le X^g$ for any $g\in G$ and $X\in\chi\}$, and for a subspace W of V, $\mathcal{I}(W)$ denotes $\{f\in k[V]|\ f(W)=0\}$. So we should be able to use these formulae along with Theorem 1.1 to construct some associated primes of cohomology modules.

Brief digression: Consider for a moment the cohomology ring $H := H^*(G,k)$ of a finite group G with coefficients in a field k whose characteristic divides the order of G. It is known (see, for example [2], Theorem 12.7.1) that the associated primes of the ring H take the form $\sqrt{\ker(\mathrm{res}^G_E)}$ for certain elementary abelian subgroups E of G. Using a result of Benson ([1], Theorem 1.1), one can show this is equal to $\sqrt{\sum_{X\in\chi} \mathrm{Tr}^G_X(H^*(X,k))}$ where χ is the set of subgroups of G not contained in any Sylow-p-subgroup of $C_G(E)$. So the associated primes of H are also radicals of relative transfer ideals. Whether this result and Theorem 1.1 are two examples of a more general phenomenon remains to be seen.

2. Annihilators in Cohomology

The following lemma is an observation of Lorenz and Pathak ([11], Lemma 1.3). It is a simple consequence of the transfer-restriction formula for cup products ([2], Theorem 4.4.2) and the starting point for our investigations. Throughout this section, let m be a strictly positive integer.

Lemma 2.1. *Suppose $\alpha\in H^m(G,R)$ satisfies $\mathrm{res}^G_N(\alpha)=0$. Then $\mathrm{Ann}_{R^G}(\alpha) \ge I^G_N$.*

[1] Here, "$<P$" means the set of all proper subgroups of P.

Proof. Let $x \in R^N$. Then we have $\mathrm{Tr}_N^G(x) \cdot \alpha = \mathrm{Tr}_N^G(x \cdot \mathrm{res}_N^G(\alpha)) = 0$. □

Corollary 2.2. *Suppose* $\alpha \in H^m(G,R)$ *and define*

$$\chi(\alpha) := \{X \le P| \ \mathrm{res}_X^G(\alpha) = 0\}. \tag{3}$$

Then $\mathrm{Ann}_{R^G}(\alpha) \ge I^G_{\chi(\alpha)}$.

Remark: Since the Sylow-p-subgroups of G are conjugate and $\mathrm{Tr}^G_{X^g}(x) = \mathrm{Tr}^G_X(xg)$, we gain nothing by considering the set of all subgroups $X \le G$ on which $\mathrm{res}^G_X(\alpha) = 0$.

The following result on annihilators is the key to proving our main theorem.

Lemma 2.3. *Let* $0 \ne \alpha \in H^m(G,R)$. *Then we have*

$$\mathrm{Ann}_{R^G}(\alpha) \le \sqrt{I^G_{<P}} = \mathcal{I}(V^P) \cap R^G.$$

Proof. The second statement is just (2) applied to the set $\{< P\}$ of all proper subgroups of P. The first is [7], Corollary 2.2, which is itself a consequence of a much more general result of Kemper ([10], Proposition 1.2). □

Lemma 2.4. *Let* $0 \ne \alpha \in H^m(G,R)$. *Then we have*

$$\sqrt{I^G_{\chi(\alpha)}} = \bigcap_{X \in \chi'(\alpha)} (\mathcal{I}(V^X)) \cap R^G$$

where $\chi'(\alpha) := \{X \le P| \ \mathrm{res}^G_X(\alpha) \ne 0\}$.

Proof. Using (2), we must show that $\chi'(\alpha)$ as defined above is equal to

$$\{X \le P| X \not\le Y^g \text{ for any } \ g \in G \text{ and } Y \in \chi(\alpha)\}.$$

This is tantamount to proving that $\mathrm{res}^G_X(\alpha) = 0$ implies $\mathrm{res}^G_{X^g} = 0$ for all $g \in G$, which is well known and follows from the fact that conjugation map $(-)^{g^{-1}}$: $X^g \to X$ induces an isomorphism $i : H^*(X,R) \to H^*(X^g,R)$ satisfying $\mathrm{res}^G_X = i \circ \mathrm{res}^G_{X^g}$. □

Our main theorem now follows from the following result:

Proposition 2.5. *Suppose* $\alpha \in H^m(G,R)$ *and* $\chi(\alpha)$ *is defined as in Corollary* 2.2. *Then*

$$\sqrt{\mathrm{Ann}_{R^G}(\alpha)} = \sqrt{I^G_{\chi(\alpha)}}$$

Remark: Suppose $\alpha \in H^1(G,k)$. Then α can be viewed as a homomorphism from G to k, which has a well-defined kernel N. Kemper ([9], Proposition 3.4)[2] proved that $\sqrt{\mathrm{Ann}_{R^G}(\alpha)} = \sqrt{I^G_N}$. For any subgroup $X \le G$, we have $\mathrm{res}^G_X(\alpha) = 0$ if and only if $X \le N^g$ for some $g \in G$. So Proposition 2.5 may be viewed as a generalisation of this result.

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[2]Kemper actually proved this result under the assumption that k is algebraically closed, although since Fleischmann's formulae (2) hold for an arbitrary field of characteristic p, the generalisation of his result to any field of characteristic p is easily obtained.

Proof. That $\sqrt{\mathrm{Ann}_{R^G}(\alpha)} \geq \sqrt{I^G_{\chi(\alpha)}}$ is an immediate consequence of Corollary 2.2. To prove the reverse, let $y^n \in \mathrm{Ann}_{R^G}(\alpha)$ for some $n \geq 0$. Let $Q \in \chi'(\alpha)$ and define $\beta := \mathrm{res}^G_Q(\alpha) \neq 0$. Then we have

$$0 = y^n \cdot \alpha = \mathrm{res}^G_Q(y^n \cdot \alpha) = y^n \cdot \beta$$

since $\mathrm{res}^G_Q : H^*(G,R) \to H^*(Q,R)$ is a ring homomorphism which specialises to the inclusion $R^G \to R^Q$ on the degree zero part. This means that $y^n \in \mathrm{Ann}_{R^Q}(\beta)$, so $y^n \in \mathcal{I}(V^Q) \cap R^Q$ by Lemma 2.3, and since this holds for every $Q \in \chi'(\alpha)$ we have

$$y^n \in R^G \cap \bigcap_{X \in \chi'(\alpha)} ((\mathcal{I}(V^X)) \cap R^X) = R^G \cap \bigcap_{X \in \chi'(\alpha)} (\mathcal{I}(V^X)) = \sqrt{I^G_{\chi(\alpha)}}$$

where the final equality follows from Lemma 2.4. Therefore $y \in \sqrt{I^G_{\chi(\alpha)}}$ as required. This completes the proof of Proposition 2.5, and since the associated primes of $H^+(G,R)$ are those annihilators of homogeneous $\alpha \in H^+(G,R)$ which are prime ideals, this completes the proof of Theorem 1.1 too.

□

References

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