Complete Information Pivotal-Voter Model with Asymmetric Group Size and Asymmetric Benefits*

Christos Mavridis[†] and Marco Serena[‡]

Abstract

We analyse a standard pivotal-voter model under majority rule, with two rival groups of players, each preferring one of two public policies and simultaneously deciding whether to cast a costly vote, as in Palfrey and Rosenthal (1983). We allow the benefit of the favorite public policy to differ across groups and impose an intuitive refinement, namely that voting probabilities are continuous in the cost of voting to pin down a unique equilibrium. The unique cost-continuous equilibrium depends on a key threshold that compares the sizes of the two groups.

1 Introduction

Consider a stylized economy with two rival groups of players; group "m", with m players, and group "n", with n players. Each player may cast a costly vote in a majoritarian election between two public policies; policy M and policy N. Players of group m receive a higher payoff if policy M is implemented, while players of group n receive a higher payoff if policy N is implemented. In a standard pivotal-voter model, we study the equilibria of

*We are grateful to Luis Corchón and Ignacio Ortuño-Ortín for their advice and support. We would like to thank the Editor, an anonymous referee, Kai Konrad, Diego Moreno and Georg Nöldeke for their useful comments as well as participants in seminars in Universidad Carlos III de Madrid, the Max Planck Institute for Tax Law and Public Finance, PET 2015 in Luxemburg and the MOMA Meeting 2016. All errors are our own.

†Department of Economics, Middlesex University London, c.mavridis@mdx.ac.uk

‡Max Planck Institute for Tax Law and Public Finance, marco.serena@tax.mpg.de

such a voting game, focusing on the case relevant for applications where both the group size and the payoffs could be different across groups.

A number of applications can be analyzed under this setup. For example, an economy made of some resource-poor and some resource-rich players is called to vote between no redistribution and full redistribution of resources, under the assumption that the poor players outnumber the rich. Each single rich player has more at stake in the election than a single poor player, as the amount of resources each individual would obtain under full redistribution is closer to the level of resources of the poor than to that of the rich. Another application is a university faculty consisting of several economists and a few lawyers, all called to vote over who to hire between two job market candidates: an economist and a lawyer.[1] Both economists and lawyers are better-off if the newly hired candidate is of their same type. Furthermore, the benefit for a lawyer from hiring another lawyer is greater than the one for an economist from hiring another economist because of the asymmetric size of the two groups; that is, since lawyers are fewer, having another lawyer in the department sharply increases each lawyer's coauthoring possibilities, whereas the benefit for an economist from having a new economist in the faculty is lower because they are already plenty. In other words, the benefit is asymmetric across players of different groups. A third application is that of residents of two neighborhoods are called to vote over the location of a new school in one of the two neighborhoods. In neighborhood N there is already a school, in neighborhood M there is none: thus, despite the fact that each resident strictly prefers the school to be located in her neighborhood, residents in neighborhood N "care less" than residents in neighborhood M about the location of the school since there is already a school in neighborhood N.

In all these applications, we may have an asymmetry in both size of each group and benefits, it is therefore an *asymmetric-asymmetric* setup which, to the best of our knowledge, has not been studied before. To study it, we adopt a complete-information

[1] The cost of voting in this case is the opportunity cost of showing up to vote that day instead of, for example, being on vacation or doing research.

pivotal-voter model with costly voting.[2] This class of models is far from novel. The seminal contribution dates back to Palfrey and Rosenthal (1983), where two groups of players each preferring one of two alternatives simultaneously decide between abstaining or voting for their preferred alternative. The winner is decided by majority rule.[3] Technical difficulties and multiplicity issues allowed Palfrey and Rosenthal to analyze only special cases.[4] The analysis of this model has been pushed forward by two other works. First, Nöldeke and Peña (2016) focus on the two groups having *symmetric* number of members and *symmetric* benefit from the favored alternative winning the election. Second, Mavridis and Serena (2018) focus on the two groups having *asymmetric* number of members and *symmetric* benefit from the favored alternative winning the election.

After analysing the *asymmetric-asymmetric* case, we deploy a continuity refinement, "cost-continuity", pinning down a unique equilibrium which we discuss. We illustrate our results focusing on the application of redistribution of resources as a running example throughout the paper, as this application has a clear and convenient parametrization of the levels of payoffs between the two groups. In particular, after describing the model in Section 2, in Section 3 we fully characterize the simple $(m; n) = (3; 2)$ model in which there are multiple equilibria.[5] We characterize analytically the equilibria where members of at least one group play a pure strategy in participation (Section 4.1, Section 4.2), and we characterize in part analytically and in part numerically the equilibria where members of both groups play a mixed strategy (Section 4.3). Our numerical analysis, which complements our analytical results, characterizes the redistribution trade-off. Despite the multiplicity of equilibria, only one equilibrium survives a novel and intuitive

[2] Our focus is different compared to other models in which there is voting over redistribution without cost of voting, as in for example Meltzer and Richard (1981). In their model, under voting, the level of redistribution is endogenously determined by the median voter and all citizens vote. We are more interested in what happens when the cost of voting is an issue, and how its magnitude affects the result of the redistribution election, in a model where voters vote only based on the possibility of being pivotal; for empirical evidence on how voters' voting decision depends on the probability of affecting the election outcome see Lyytikäinen and Tukiainen (2019).

[3] I.e., the alternative that gets more than 50% of the votes wins.

[4] To see these special cases, we refer to footnote 1 in Mavridis and Serena (2018).

[5] Palfrey and Rosenthal (1983) already noticed that the asymmetric-symmetric model suffers from a multiplicity issue.

cost-continuity refinement; that is, the equilibrium probability of voting is continuous in the cost of voting. In fact, our numerical analysis shows that the cost-continuity refinement turns out to single out a unique equilibrium in the general model with an arbitrary number of poor and rich players (see Section 5). We study the properties of the unique cost-continuous equilibrium; if the size of the poor group is sufficiently small (namely, the number of poor players is *lower* than the square of the number of the rich players), then the poor players may vote and redistribution has a chance of winning, while if the size of the poor group is large (namely, the number of poor players is *greater* than the square of the number of the rich players), the unique cost-continuous equilibrium dictates that poor players abstain with certainty, and thus poor players are doomed to lose the election and redistribution is not implemented.

The above threshold suggests an answer to the redistribution trade-off. Consider an increase in the number of poor players. On the one hand, it makes the individual resources under redistribution closer to the resources of a poor and further apart from those of a rich. This makes the stakes of the rich in the election increase, and that of the poor decrease.[6] On the other hand, an increase in the number of poor players makes the poor group bigger and thus stronger in the election; that is, if all players were to vote with the same exogenous probability full redistribution would be the most likely outcome of the election as there are more poor players than rich. All in all, the former effect of an increase in the number of poor players (making rich players vote "more") is *stronger* than the latter effect (making rich players relatively less numerous) the greater is the number of poor players; in fact, when the number of poor players is greater than the square of the number of the rich players, the former dominates the latter and in the unique cost-continuous equilibrium poor players have no chance of winning the election. The opposite happens when the number of poor players is low, and hence poor players have a chance of democratically redistributing resources.

[6] Acemoglu and Robinson (2000, 2012) examine a similar idea, in which if there are many poor players in a society the threat of revolution decreases since the spoils would be divided over a larger mass of players.

After their 1983 paper, Palfrey and Rosenthal (1985) analyzed a version of the model under private information on the cost of voting and from then onwards, much of the literature has developed under private information, as the model is often more tractable and allows more elegant and neat results (e.g., Borgers, 2004; Taylor and Yildirim, 2010). Nevertheless, the general idea that voters compare the benefits of voting with its costs is older and has been long the interest of economists, dating back at least to Downs' (1957) seminal work. We consider only benefits that accrue from changing the policy to the voters' preferred outcome, despite a number of other benefits playing an important role in real-life; for instance, Wiese and Jong-A-Pin (2017) empirically examine the benefits arising from "expressive" motives of voting. In our model, we follow the main strand of the literature on complete-information pivotal-voter model with costly voting, started by Palfrey and Rosenthal (1983), and further developed by Nöldeke and Peña (2016) and Mavridis and Serena (2018). And as we generalize this strand of the literature to the asymmetric-asymmetric setup described above, we abstract away from other interesting forces that typically play an important role in real life voting. An example is communication. In the laboratory, Palfrey and Pogorelskiy (2019) find that communication increases turnout when messages are public within one's own party, and decreases turnout with a low voting cost when subjects exchange public messages through computers. In the analysis of correlated equilibria, Pogorelskiy (2020) finds that communication helps sustain equilibria with high levels of turnout. Kalandrakis (2007, 2009), under heterogeneous voting costs, finds that almost all Nash equilibria are robust to small amounts of incomplete information. Social pressure is also a key aspect of voters' turnout which we abstract away from (see e.g. Gerber et al., 2008; Gerber et al., 2016; and DellaVigna et al., 2017). Finally, polls are known to have a significant effect on voters' turnout (e.g., Großer et al., 2010; Morton et al., 2015).

2 Model

There are two types $i \in \{m, n\}$ of players; $m > 1$ poor players and $n > 1$ rich players. Assume poor players are more numerous; $m > n$.[7] Players are simultaneously called to cast a vote between two alternatives, M and N. When we specialize the model to redistribution of resources, M is full redistribution and N is no redistribution of resources. Poor players prefer alternative M, in that if M wins (rather than loses) the individual payoff of a poor player increases by $\Delta\pi_m > 0$. Symmetrically, rich players prefer alternative N, in that if N wins (rather than loses) the individual payoff of a rich player increases by $\Delta\pi_n > 0$. Players choose whether to vote for their preferred alternative or to abstain, since voting for the non-preferred alternative is strictly dominated. If a player of type $i \in \{m, n\}$ casts a vote, she faces a cost of voting $c_i > 0$. Thus, the increase in payoff –net of cost of voting– for a player i when her preferred alternative wins is $\Delta\pi_i - c_i$ if she voted, and $\Delta\pi_i$ if she did not vote. Players vote simultaneously and the winning alternative is decided by majority rule. Ties are broken by a fair coin toss.

(Redistribution-Parametrized Example) Throughout the paper, our running example is that of resource redistribution, where each poor player has an initial amount of resources equal to 1 and each rich player has an initial amount of resources equal to 2.[8] Under full redistribution each player ends up with the average resources in the economy, which is:

$$\frac{2n+m}{n+m}$$

and under no redistribution everyone keeps her original amount of resources. Thus,

$$\Delta\pi_m = \frac{2n+m}{n+m} - 1 = \frac{n}{n+m} \quad (1)$$

$$\Delta\pi_n = 2 - \frac{2n+m}{n+m} = \frac{m}{n+m} \quad (2)$$

[7] By assuming $m > n > 1$ we avoid having to deal with trivial case distinctions of $m = n$ or $n = 1$ throughout the paper.

[8] It will be clear that these resource level assumptions are qualitatively without loss of generality.

Since the resources of the poor is less than the average resources, a poor m player would always prefer full redistribution (alternative M) and a rich n player would always prefer no redistribution (alternative N). Furthermore, notice from (1) and (2) that $\Delta\pi_n > \Delta\pi_m$; that is, a single rich player has more at stake in the election than a single poor player.[9]

We denote by p_i the probability of voting of player i.[10] This probability maximizes her individual expected payoff, taking as given the choices of the other players. As in Palfrey and Rosenthal (1983), we consider Quasi-Symmetric Nash Equilibria (QSNE), that is, players of group i follow the same equilibrium strategy p_i^*. Hence, a QSNE is a pair (p_i^*, p_j^*) such that a player of group $i \in \{m, n\}$ would not want to deviate from p_i^* if she expects every other player of group i to also play p_i^* and all players of group j with $j \neq i$ to play p_j^*. A QSNE can be of one of the following three types:

1. "Pure": $(p_m^*, p_n^*) \in \{0, 1\}^2$
2. "Partially Mixed": $p_m^* \in \{0, 1\}, p_n^* \in (0, 1)$ or $p_m^* \in (0, 1), p_n^* \in \{0, 1\}$
3. "Totally Mixed": $(p_m^*, p_n^*) \in (0, 1)^2$.

Our asymmetric-asymmetric setting (see Introduction) will allow us to fully characterize the first two types of QSNE, and for sufficiently low (m, n) also the third. However, for arbitrary (m, n) we will tackle the characterization of the "Totally Mixed" equilibria partly analytically and partly numerically.

Define A_i to be the probability that the vote of a player of group i is pivotal. A player of group i may cast a vote if:

$$A_i \frac{\Delta\pi_i}{2} \geq c_i. \tag{3}$$

[9]Considering the extrema of no- and full-redistribution is, to some extent, without loss of generality. Since the resources of the poor are less than the average resources, the poor would like as much redistribution as possible and, at the same time, the rich would like as little redistribution as possible. Therefore, given a choice between any two proportional tax rates schemes, the poor would want to vote for the higher one and the rich for the lower one. In the end, there is no loss of generality to assume that the two tax rates that are competing are 0 (no redistribution) and 1 (full redistribution).

[10]Note that p_i is a function of c, m, and n. However, for brevity, we omit its dependence on m and n.

Conditional on being pivotal, the extra utility of creating a tie (from 0 to $\frac{\Delta\pi_i}{2}$) or breaking a tie (from $\frac{\Delta\pi_i}{2}$ to $\Delta\pi_i$) is identical since the tie breaking rule is fair. This property explains the division by 2 in (3) and holds in general throughout the paper. The above inequality is identical to

$$A_i \geq \frac{2c_i}{\Delta\pi_i} \equiv B_i \tag{4}$$

for $i \in \{m, n\}$.

(Redistribution-Parametrized Example) In our redistribution parametrized example, we set for simplicity the costs of voting identical across types, $c_m = c_n = c$. Then, given the specifications (1) and (2), we can write

$$B_m = \frac{2c(n+m)}{n}, \tag{5}$$

$$B_n = \frac{2c(n+m)}{m}, \tag{6}$$

so that a poor player may cast a vote if $A_m \geq \frac{2c(n+m)}{n}$ and a rich player if $A_n \geq \frac{2c(n+m)}{m}$. Notice that $B_m > B_n$; as discussed in the Introduction, a rich individual rich has more at stake in the election than a poor individual.

In what follows we start the analysis focusing first on an easy non trivial case: $(m, n) = (3, 2)$. Then, we characterize the general (m, n) case.

3 $(m, n) = (3, 2)$ in the *(Redistribution-Parametrized Example)*

We start with a simple example where we assume that $m = 3$ and $n = 2$ in our *(Redistribution-Parametrized Example)*.[11] A single player will analyze all the possible

[11] The case of $n = 1$ is interesting, but it has very unique properties. The equilibrium is unique without needing for a refinement. Nevertheless, we deem the illustration of the case of a single poor individual not

scenarios of what the other four players will do.

Focus on a poor player first. A single poor player knows that there are nine possible cases depending on whether the other four players (two rich and the other two poor players) vote or abstain. We consider the nine cases in the following table where $(\bar{m}, \bar{n})$ represent the number of players from each group that turn out to vote; that is, if $(\bar{m}, \bar{n}) = (0, 1)$, the other poor players do not vote, while one rich does and the other rich does not.

Poor player: pivotal if voting?	$(\bar{m}, \bar{n})$	Probability	$\frac{\Delta\pi_i}{2}$
Yes	(2,2)	$p_m^2 p_n^2$	$\frac{1}{5}$
Yes	(1,1)	$4p_m(1-p_m)p_n(1-p_n)$	$\frac{1}{5}$
Yes	(0,0)	$(1-p_m)^2(1-p_n)^2$	$\frac{1}{5}$
Yes	(0,1)	$2(1-p_m)^2 p_n(1-p_n)$	$\frac{1}{5}$
Yes	(1,2)	$2p_m(1-p_m)p_n^2$	$\frac{1}{5}$
No	(2,0)	$p_m^2(1-p_n)^2$	0
No	(2,1)	$2p_m^2 p_n(1-p_n)$	0
No	(1,0)	$2p_m(1-p_m)(1-p_n)^2$	0
No	(0,2)	$(1-p_m)^2 p_n^2$	0

In this table, every row corresponds to each possible case: the first three rows correspond to the cases where the vote of the poor player would break a tie. The fourth and fifth rows correspond to the case where her vote would create a tie. The last four rows correspond to the case where her vote would make no difference to the outcome of the election. The column Probability gives the probability that each case realizes. For example the $(2, 1)$ case realizes if the other two poor vote, one rich votes and the other rich does not. The single poor player examining this table knows that she is pivotal in fives cases out of nine. If she is not pivotal she would rather not vote, so as to save on the cost c. If she is pivotal she may want to vote, provided that the net utility from voting is positive.

relevant for three reasons. First, it helps the reader neither to understand what the equilibrium analysis for $n > 1$ yields nor to observe the multiplicity of equilibria which is key in our setup. Second, the case of a single rich individual is, in our opinion, not relevant for the applications we want to encompass. Third, all the forces due to the need of within-group coordination among individuals of the same type (rich) are absent when there is only one member of the rich type. In other words, rich perfectly coordinate and never free-ride on each others because there is only one rich individual.

Following the table above and the voting condition (3), the necessary condition for a poor player to be willing to pay the cost of voting is:

$$[p_m^2 p_n^2 + 4p_m(1-p_m)p_n(1-p_n) + (1-p_m)^2(1-p_n)^2 + \\ 2(1-p_m)^2 p_n(1-p_n) + 2p_m(1-p_m)p_n^2]\frac{1}{5} \geq c, \quad (7)$$

where the $\frac{\Delta\pi_i}{2} = \frac{1}{5}$ because;

1. if the poor votes and breaks a tie, she obtains the full redistribution outcome 7/5 while not voting would give her the tie outcome $(1/2)\,(7/5) + (1/2)\,1$,

2. if the poor votes and creates a tie, she obtains the tie outcome $(1/2)\,(7/5) + (1/2)\,1$ while not voting would give her the no redistribution outcome 1,

and in both cases the difference in payoff is 1/5.

A rich player solves a similar problem, considering the six corresponding cases. The resulting condition for the rich to want to vote is:

$$[3p_m^2(1-p_m)p_n + 3p_m(1-p_m)^2 p_n + 3p_m(1-p_m)^2(1-p_n) + (1-p_m)^3(1-p_n)]\frac{3}{10} \geq c. \quad (8)$$

When looking for an equilibrium, note that its nature could be any between Pure, Partially Mixed and Totally Mixed. For instance, in order to sustain a Pure equilibrium with $p_m^* = p_n^* = 0$, it needs to be the case that $A_i \leq B_i$ for both groups (in other words, the direction of (7) and (8) is inverted) and recall that A_i depends on p_m and p_n themselves. When instead we investigate the existence of Partially Mixed equilibria of the form $p_m^* \in (0,1)$ and $p_n^* = 1$, then the n-individuals must be better-off voting than abstaining, and hence we need $A_n \geq B_n$, whereas the m-individuals must be indifferent between voting and abstaining, and hence we need $A_m = B_m$. Once again, $p_n^* = 1$ is then plugged into $A_m = B_m$, where the latter equation gives a (possibly unique) p_m^*, which then needs to satisfy both $p^*_m \in (0,1)$ as well as $A_n \geq B_n$. More completely, if a pair (p^*_m, p^*_n) leads to Inequality 7 (8) to hold strictly then it must be that $p^*_m = 1$ ($p^*_n = 1$). If a pair (p^*_m, p^*_n) leads to Inequality 7 (8) to not hold then it must be that $p^*_m = 0$ ($p^*_n = 0$). These two cases form the four possible "Pure" QSNE; $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. If a pair (p^*_m, p^*_n) leads to the two inequalities to hold with equality, then the pair (p^*_m, p^*_n) forms a "Totally Mixed" QSNE. If a pair (p^*_m, p^*_n) leads Inequality 7 to hold with equality, and Inequality 8 to hold with $<$ ($>$) then the pair is an equilibrium if $p^*_n = 0$ ($p^*_n = 1$). If a pair (p^*_m, p^*_n) leads Inequality 8 to hold with equality, and Inequality 7 to hold with $<$ ($>$) then the pair is an equilibrium if $p^*_m = 0$ ($p^*_m = 1$). These last two cases form "Partially Mixed" QSNE. We proceed in the above described way for all possible equilibria. In the simple example of this section, the sustainability and full characterization of all equilibria can be performed algebraically. In the more complex case of Section 5, we use numerical methods, but proceed in the same manner.

Such reasoning leads to the general solution depicted in Figure 1, where we adopted the notation consistent with our *(Redistribution-Parametrized Example)*; namely, $B_n = B$ and $B_m = \frac{m}{n}B$, so as to have only one parameter B simplifying the graphical exposition (see (5) and (6)).[12]

For any B, we examine all the types of equilibria: "Pure", "Partially Mixed" and "Totally Mixed". We find that for low B there are three different types of equilibria, which are depicted in the first row of Figure 1, while for larger B we only have the equilibrium depicted in the second row. The unique equilibrium for B sufficiently large corresponds to the characterization in Propositions 1 to 5 (Subsections 4.1 and 4.2).

Note that, importantly, it is challenging to provide a thorough intuition behind the equilibria. This issue is common in the costly voting pivotal-model literature—see Palfrey and Rosenthal (1983) and Nöldeke and Peña (2016); most features of the equilibria are driven by indifference condition of mixed-strategy equilibria.

The maximum values of B for existence of each equilibrium in the first row are $B' =$

[12]In this $(m,n) = (3,2)$ case, this notation implies $B = \frac{10}{3}c$.

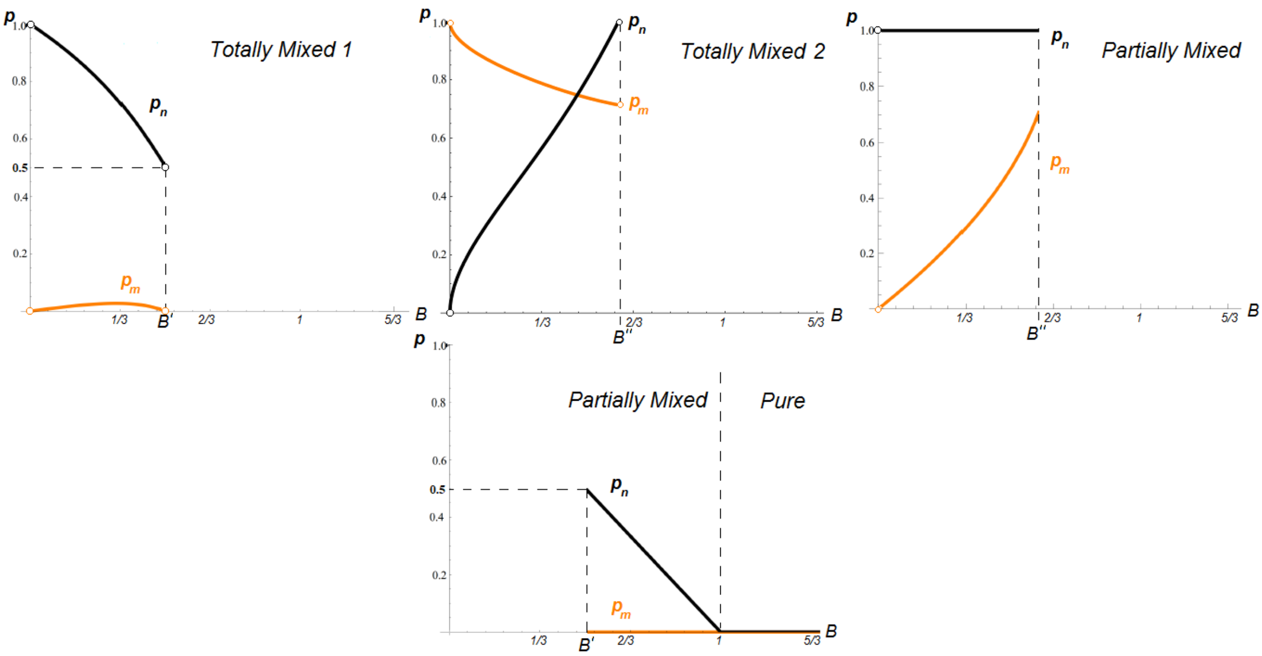

Figure 1: The four panels present all the equilibria for the $(3, 2)$ example. In the first row, first panel, the "Totally Mixed 1". In the first row, second panel, the "Totally Mixed 2". In the first row, third panel, the "Partially Mixed". In the bottom panel, "Partially Mixed" continuously connected to the "Pure".

1/2, $B'' = 30/49$, $B'' = 30/49$ respectively. The minimum value of B for existence of the equilibrium in the second row is $B' = 1/2$. Now, take for example $B = 1/3$. There are three equilibria: respectively, one in which both types of players are playing a mixed strategy , with the rich voting with a much higher probability than the poor; one that both are playing a mixed strategy but now the poor are voting with higher probability than the rich; and one that the rich vote for sure and the poor play a mixed strategy. On the other hand for larger B (but not too large), say $B = 2/3$, there is only one type of equilibrium, the one where the rich are playing a mixed strategy, and the poor abstain for sure. And, naturally, if B is very high (i.e., greater than 1), the cost of voting is much greater than the benefit, thus no-one votes.[13]

Cost-continuity. Beginning from a high B we will have to be on the "Pure" equilibrium in the bottom panel of the Figure. As B decreases and gets lower than 1, a unique equilibrium exists which is the "Partially Mixed" equilibrium which is drawn in the same panel. As we keep decreasing B, we move along the two curves of the "Partially Mixed" equilibrium until we reach $B'' = 30/49$, where the "Totally Mixed 2" equilibrium (second panel) and the Partially Mixed equilibrium in which the rich are voting with certainty (third panel) start being defined; however moving to any of these equilibria would involve an upwards jump in both probabilities of voting. However, exactly where the "Partially Mixed" equilibrium of the fourth panel stops being defined, is where the "Totally Mixed 1" equilibrium starts being defined, and on top of that the probabilities change continuously from one equilibrium to the other with a value of 0.5 We would expect a small decrease in costs or benefits to incur a reasonably small effect on the probabilities of voting. For this reason, we consider the smooth transition of the unique cost-continuous equilibrium more plausible than the discontinuous behavior of the other two equilibria, and thus our cost-continuity refinement selects the "Totally Mixed 1" together with the equilibria in the fourth panel.

We can use a voting station analogy to explain the reasoning of the above. If the voting

[13]The probability of being pivotal is at most 1, and thus the voting condition (4) is never satisfied.

station is in another continent the unique equilibrium is that no citizen finds it worthy to vote. As the voting center is moved closer to the citizens, the cost of voting decreases. At some point the voting center will be close enough that the unique equilibrium is that the rich find it reasonable to vote with positive probability. The point of the cost-continuity refinement is that if the voting center station moves slightly closer to a citizen, we should expect her equilibrium probability of voting to also increase slightly. While the poor may still find it too expensive to vote, the equilibrium voting probability of the rich continues to increase as the voting station comes closer. At some point, when the voting center is close enough and given the rich probability of voting, the poor voters find it profitable to vote as well. As the voting center comes even closer, both the rich and poor equilibrium probabilities of voting change continuously.

Formally, we use a standard definition of continuity for all $c > 0$, or $B > 0$ (see for example page 943 of Mas-Colell et al. 1995). This definition means that, for all $i, j \in \{m, n\}$, $i \neq j$, there is a single continuous selection of the equilibrium correspondences p_i mapping (B_i, B_j) to equilibrium probabilities of voting.

Numerical simulations in Section 4.3, building on the analytical results of sections 4.1-4.3, will show that the uniqueness of a cost-continuous equilibrium is a property holding, not only for $(m, n) = (3, 2)$, but for general (m, n). We will focus on the unique cost-continuous equilibrium and run comparative statics to shed light on the redistribution trade-off spelled out in the Introduction and on how it depends on the number of poor and rich players in the economy. We will also provide the intuition for our results, and derive a group-size threshold describing when full redistribution has or does not have a chance of winning the election.

4 General (m, n)

The probability of being pivotal is crucial for the voting/abstention choice in (4). While we discussed it in the previous section for the special case of $(m, n) = (3, 2)$, it is useful to

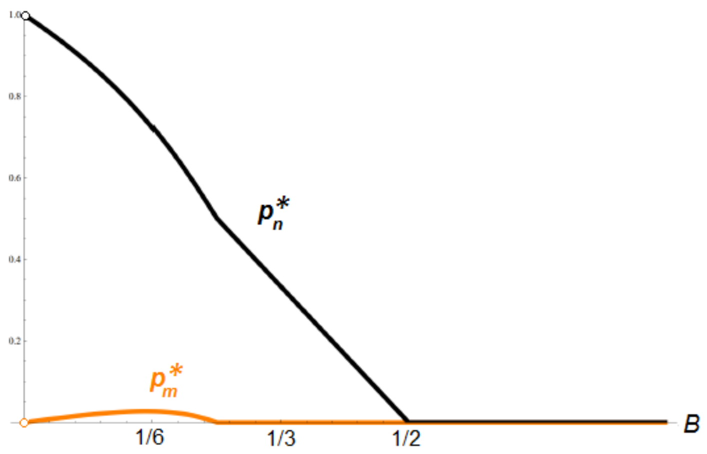

Figure 2: Unique continuous equilibrium for $m = 3$ and $n = 2$.

provide the expression for a general pair (m, n). The following table shows the calculations that a single poor player must make when she knows that she faces $m-1$ poor players and n rich with $(\bar{m}, \bar{n})$ signifying how many of the rest actually vote in each instance.

Poor player: pivotal if voting?	$(\bar{m}, \bar{n})$	Probability	$\Delta\pi_i$
Yes	(n, n)	$\frac{(m-1)!}{n!(m-1-n)!}p_m^n(1-p_m)^{m-n-1}p_n^n$	$\frac{n}{n+m}$
Yes	$(n-1, n)$	$\frac{(m-1)!}{(n-1)!(m-n)!}p_m^{n-1}(1-p_m)^{m-n}p_n^n$	$\frac{n}{n+m}$
Yes	$(n-1, n-1)$	$\frac{(m-1)!}{(n-1)!(m-n)!}np_m^{n-1}(1-p_m)^{m-n}p_n^{n-1}(1-p_n)$	$\frac{n}{n+m}$
Yes	$(n-2, n-1)$	$\frac{(m-1)!}{(n-2)!(m-n+1)!}np_m^{n-2}(1-p_m)^{m-n+1}p_n^{n-1}(1-p_n)$	$\frac{n}{n+m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Yes	$(0, 1)$	$n(1-p_m)^{m-1}p_n(1-p_n)^{n-1}$	$\frac{n}{n+m}$
Yes	$(0, 0)$	$(1-p_m)^{m-1}(1-p_n)^n$	$\frac{n}{n+m}$
No	All other cases		0

For economy of space we have omitted the cases that the poor player is not pivotal. The resulting expression for the pivotal probability is as follows:

$$A_i = \sum_{s=0}^{n} \binom{i-1}{s}\binom{j}{s} p_i^s(1-p_i)^{i-s-1}p_j^s(1-p_j)^{j-s} + \sum_{s=0}^{n-1} \binom{i-1}{s}\binom{j}{s+1} p_i^s(1-p_i)^{i-s-1}p_j^{s+1}(1-p_j)^{j-s-1} \tag{9}$$

for $i, j \in \{m, n\}$, $i \neq j$.

With the general expression for A_i in our hands, we can analyse the QSNE of the general voting game, given the relative costs of voting B_i and B_j, and classify them according to their type: "Pure", "Partially Mixed" or "Totally Mixed". As we will see, the first two types (i.e., equilibria in which some players play pure strategies) can be fully characterized, as we do in Subsections 4.1 and 4.2, while the last type of equilibria will be tackled in Subsection 4.3.

4.1 "Pure" equilibria and "Partially Mixed" equilibria with abstention

If $A_i < B_i$, player i's dominant strategy is to abstain. On the other hand if $A_i > B_i$ player i's dominant strategy is to vote. When $A_i = B_i$ player i is indifferent between voting and abstaining; therefore this is a necessary condition for player i to employ a mixed strategy. From this condition we see that B_i can be interpreted as the minimum A_i for player i to be willing to vote. Clearly if $B_i > 1$, player i abstains.

Proposition 1. *If* $B_i \geq 1$ *then* $p_i^*(B_i) = 0$.

Proof. Trivial generalization of Proposition 1 in Mavridis and Serena (2018). □

The previous proposition shows that if relative costs of voting B_i are high enough for players of both groups, the only equilibrium that exists is the "Pure" one in which nobody votes.

Obviously, the situation described in Proposition 1 is not very interesting. Therefore, next we allow one of the two relative costs of voting to be low enough such that players from one of the two groups might consider voting; in other words, $B_i \geq 1$ only for players i. Then Proposition 2 yields a simple and unique characterization of j's equilibrium strategy:

Proposition 2. *For $B_i \geq 1$ and $B_j \in (0,1)$ the unique QSNE is that $p_i^*(B_i) = 0$ and $p_j^*(B_j) = 1 - B_j^{\frac{1}{j-1}}$, for all $i, j \in \{m, n\}$, $i \neq j$.*[14]

Proof. By Proposition 1, $p_i^* = 0$. Suppose $p_j^* = 0$. Then any single j player would have an incentive to deviate and vote for sure in order to single-handedly decide the election in favor of the j-group. Thus $p_j = 0$ is not an equilibrium. On the other hand, suppose $p_j^* = 1$. This means that j group wins for sure with a margin of j votes. Then any single j-player would have an incentive not to pay the cost without affecting the outcome. Thus $p_j = 1$ is not an equilibrium. Therefore $p_j^* \in (0,1)$. Plugging $p_i^* = 0$ in A_j (see Equation 9) we have $A_j = (1-p_j)^{j-1}$, and since (4) must hold with equality for players j to mix, we have:

$$(1-p_j)^{j-1} = B_j$$

or equivalently:

$$p_j = 1 - B_j^{\frac{1}{j-1}}.$$

□

Note that this proposition is not paralleled by any result in Mavridis and Serena (2018), which assume that $B_i = B_j$.

The two previous propositions examine cases in which players from at least one group find it too costly to vote, no matter what players from the other group do. These cases gave rise to two types of equilibria; a "Pure" equilibrium in which everybody's strictly dominant strategy is not to vote (Proposition 1), and a "Partially Mixed" equilibrium in which players from one group have a strictly dominant strategy not to vote and players from the other group play a mixed strategy (Proposition 2).

Comparative Statics. For $x \in (0,1)$ the expression $1 - x^{\frac{1}{j-1}}$ is strictly decreasing in x. Therefore $p_j^*(B_j) = 1 - B_j^{\frac{1}{j-1}}$ is strictly decreasing in c_j and strictly increasing in $\Delta\pi_j$. Higher individual cost-payoff ratio results in j-players voting with lower probability. [15]

[14] As a reminder, in the beginning of this section we have assumed $m > n > 1$. It is easy to see that for $j = 1$ the unique equilibrium is: $p_i^* = 0$ and $p_j^* = 1$. We write p_i^* rather than $p_i^*(B_i)$ in all proofs, for brevity.

In the remainder of this subsection we examine what happens when players from neither group have a strictly dominant strategy to abstain, ie. what happens when $B_m < 1$ and $B_n < 1$. Under these conditions players of both groups may vote with positive probability. This causes strategic interactions that may generate multiple equilibria.

It is easy to see that when $B_m < 1$ and $B_n < 1$ no "Pure" equilibria exist.

Proposition 3. *For $B_i < 1$ and $B_j < 1$, no "Pure" QSNE exist, for all $i, j \in \{m, n\}$ and $i \neq j$.*

Proof. Trivial generalization of Proposition 2 in Mavridis and Serena (2018). □

After proving that for $B_m < 1$ and $B_n < 1$ no "Pure" equilibria exist, the next proposition establishes that for $B_m < 1$ and $B_n < 1$ a "Partially Mixed" equilibrium does exist.

Proposition 4. *For $B_i < 1$ and $B_j < 1$, there exists a "Partially Mixed" QSNE with $p_i^*(B_i) = 0$ and $p_j^*(B_j) = 1 - B_j^{\frac{1}{j-1}}$, for all $i, j \in \{m, n\}$ and $i \neq j$ if and only if $B_i \geq \underline{B_i} \equiv jB_j - (j-1)B_j^{\frac{j}{j-1}}$.*

Proof. An equilibrium where $p_i^* = 0$ implies: $A_j = (1-p_j)^{j-1}$ and $A_i = (1-p_j)^j + jp_j(1-p_j)^{j-1}$. The former means that a player j is pivotal only if none of her groupmates vote (her vote breaks the tie in which nobody votes). The latter means that a player i is pivotal if none of j players vote or if only one of them votes. In order for the i-players to not want to vote we must have:

$$A_i \leq B_i,$$

or equivalently

$$(1-p_j)^j + jp_j(1-p_j)^{j-1} \leq B_i,$$

and similarly, for the j-group player to play a mixed strategy we must have:

$$(1-p_j)^{j-1} = B_j, \tag{10}$$

[15] See also Kalandrakis (2007) who focusing on monotone equilibria, finds a similar comparative statics.

dividing the two conditions and rearranging we get:

$$\begin{aligned} 1 - p_j + jp_j &\leq \frac{B_i}{B_j} \\ (j-1)p_j &\leq \frac{B_i}{B_j} - 1 \end{aligned} \tag{11}$$

Isolate p_j in (10) and plug it in (11) to get

$$(j-1)\left(1 - B_j^{\frac{1}{j-1}}\right) \leq \frac{B_i}{B_j} - 1 \tag{12}$$

Or equivalently,

$$\begin{aligned} -B_j^{\frac{j}{j-1}} &\leq \frac{B_i - jB_j}{j-1} \\ B_i &\geq jB_j - (j-1)B_j^{\frac{j}{j-1}} \equiv \underline{B_i} \end{aligned} \tag{13}$$

□

$\underline{B_i}$ is an increasing bijection from $[0,1]$ to $[0,1]$, such that if $B_j = 0$, $\underline{B_i} = 0$, and if $B_j = 1$, $\underline{B_i} = 1$.

Note that the equilibria pinned down by Proposition 2 and Proposition 4 are essentially the same, the difference being that Proposition 2 provides the range of B_i's for which the equilibrium is unique, and Proposition 4 provides the range of B_i's for which that equilibrium continues to exist although not necessarily uniquely. This is an important finding in terms of uniqueness of a cost-continuous equilibrium. In fact, first notice that Proposition 4 gives us for every $B_j \in (0,1)$ the lowest B_i for existence of the "Partially Mixed" equilibrium $p_i^*(B_i) = 0$ and $p_j^*(B_j) = 1 - B_j^{\frac{1}{j-1}}$. This satisfies the system $A_i \leq B_i$ and $A_j = B_j$, which implies that the system $A_i = B_i$ and $A_j = B_j$ is also satisfied (A_i's are continuous in p_i's). Therefore, exactly at those values of (B_i, B_j) there must be a "Totally Mixed" equilibrium. In our *(Redistribution-Parametrized Example)*, as discussed in Section 3 we set $B_n = B$ and $B_m = \frac{m}{n}B$. Numerical simulations show that that "Totally Mixed" equilibrium survives from that value of B all the way down to 0.

Proposition 4 is silent with respect to which of the two groups will be playing a mixed strategy and which will not be voting. What it says is that if $B_m < 1$ and $B_n < 1$ it can be either that the m players do not vote and the n players play a mixed strategy, or that the n players do not vote and the m players play a mixed strategy. The next proposition shows that for a given pair (B_i, B_j) these two "Partially Mixed" equilibria of Proposition 4 cannot co-exist; in other words, for a given pair (B_i, B_j) we either have m players not voting and n playing a mixed strategy *or* n players not voting and m playing a mixed strategy (but not both).

Proposition 5. *For $B_i < 1$ and $B_j < 1$, $B_i \geq \underline{B_i}$ and $B_j \geq \underline{B_j}$ are mutually exclusive, for all $i, j \in \{m, n\}$ and $i \neq j$.*

Proof. Suppose not and consider the $(B_i, B_j)-$space. We first show that $\underline{B_i} > B_j$, or equivalently:

$$\begin{aligned} (j-1)B_j &> (j-1)B_j^{\frac{j}{j-1}} \\ 1 &> B_j^{\frac{1}{j-1}}. \end{aligned}$$

For the same reason we also have $\underline{B_j} > B_i$. Then, $B_i \geq \underline{B_i}$ and $\underline{B_i} > B_j$ imply $B_i > B_j$, while $B_j \geq \underline{B_j}$ and $\underline{B_j} > B_i$ imply $B_j > B_i$ leading to a contradiction. □

4.2 "Partially Mixed" equilibria with voting

We are left to analyze the "Partially Mixed" equilibria in which players of one group are voting with certainty and the others are playing a mixed strategy. There cannot be an equilibrium in which the majority group votes with certainty and the minority group plays a mixed strategy (see Mavridis and Serena (2018) for a proof). Therefore the equilibria left to analyze are of the type $p_n^* = 1$ and $p_m^* \in (0,1)$. The next four propositions will establish conditions for this type of equilibrium to exist, and Proposition 10 will characterize the equilibria behavior.

Proposition 6. *There exists a* $\hat{B} < 1$ *such that if* $B_m \leq B_n \leq \hat{B}$ *holds there exists a unique "Partially Mixed" equilibrium of the form* $p_n^*(B_n) = 1$ *and* $p_m^*(B_m) \in (0,1)$.

Proof. It follows from Proposition 3 in Mavridis and Serena (2018). However we repeat some key steps of their proof that will help the proofs of the following propositions.

When $p_n^* = 1$ we have:

$$A_m = \binom{m-1}{n} p_m^n (1-p_m)^{m-n-1} + \binom{m-1}{n-1} p_m^{n-1} (1-p_m)^{m-n}$$

and

$$A_n = \binom{m}{n-1} p_m^{n-1} (1-p_m)^{m-n+1} + \binom{m}{n} p_m^n (1-p_m)^{m-n}.$$

A_m is strictly increasing in p_m in $(0, \hat{p}_m)$ and strictly decreasing otherwise, with

$$\hat{p}_m = \frac{n(n-1)}{n(n-1) + \sqrt{n(n-1)(m-n)(m-n-1)}}.$$

We furthermore know that $A_m \geq A_n$ if and only if $p_m \leq p_m^{**}$, with

$$p_m^{**} = \frac{n(n-1)}{n(n-1) + \sqrt{n(n-1)(m-n)(m-n+1)}}.$$

The cut-off for costs is therefore $\hat{B} = A_m|_{p_m = p_m^{**}}$. □

The following proposition shows that an equilibrium may not exist if either cost is too high.

Proposition 7. *If* $B_n > \hat{B}$ *or if* $B_m > B^{max}$, $B^{max} = A_m|_{p_m=\hat{p}_m}$, *no "Partially Mixed" equilibrium of the form* $p_n^*(B_n) = 1$ *and* $p_m^*(B_m) \in (0,1)$ *exists.*

Proof. We will prove that A_n is strictly increasing in p_m for $p_m < p_m^{**}$ and strictly decreasing for $p_m > p_m^{**}$. Take the derivative of A_n with respect to p_m. This derivative is greater than zero when:

$$\binom{m}{n-1}((n-1)p_m^{n-2}(1-p_m)^{m-n+1} - (m-n+1)p_m^{n-1}(1-p_m)^{m-n}) + \binom{m}{n}(np_m^{n-1}(1-p_m)^{m-n} - (m-n)p_m^{n}(1-p_m)^{m-n-1}) \geq 0$$

Simplifying this expression we get:

$$m(m-2n+1)p_m^2 + 2pn(n-1) - n(n-1) \leq 0,$$

which is precisely Inequality (5) of Mavridis and Serena (2018). Using a similar analysis as in that paper we see that for $p_m < p_m^{**}$ A_n is strictly increasing and for $p_m > p_m^{**}$ it is strictly decreasing. This implies that A_n is strictly increasing when $A_n > A_m$ and strictly decreasing when $A_n < A_m$, which further implies that the point of intersection between A_m and A_n is the maximum of A_n. Therefore, $\hat{B} = A_m|_{p_m = p_m^{**}}$, and if $B_n > \hat{B}$ this would necessarily imply $A_n < B_n$. A similar argument follows for $B_m > B^{max}$, $B^{max} = A_m|_{p_m = \hat{p}_m}$. □

Naturally if the cost of voting is too high for any group then nobody from that group will vote, and no "Partially Mixed" equilibrium of the form $p_i^* = 1$ and $p_j^* \in (0,1)$ exists. The ranking between B_n and B_m on its own does not affect the existence of this type of equilibrium, but may affect its uniqueness. While Proposition 6 shows the existence and uniqueness of an equilibrium when $B_n \geq B_m$ (provided of course that $\hat{B} \geq B_n$), the next two propositions show that if $B_m \geq B_n$ the equilibrium exists but may no longer be unique; there might be two equilibria of this kind.

Proposition 8. *Let $m = n+1$ and $B^{max} > B_m > B_n$ with $\hat{B} > B_n$. Then there exists a unique "Partially Mixed" equilibrium given by $p_n^*(B_n) = 1$ and $p_m^*(B_m) \in (0,1)$ if*

$A_n|_{p_m=p_m^*(B_m)} \geq B_n$. Let $m = n+1$ and $B_m = B^{max}$, then there is no "Partially Mixed" equilibrium.

Proof. The first part of the proof follows from Proposition 3 in Mavridis and Serena (2018). Notice that if $n = m-1$ then $\hat{p}_m = 1$ and in this case A_m is strictly increasing for $p_m \leq 1$. Because of this $A_m = B_m$ has a unique solution $p_m^* > p_m^{**}$. The equilibrium exists as long as $A_n|_{p_m=p_m^*} \geq B_n$. For the second part notice that when $B_m = B^{max}$ then necessarily $p_m^* = 1$. But this means that $A_n|_{p_m=1} = 0$, which means that nobody from the minority group would vote. □

The next proposition considers the $m > n+1$ case.

Proposition 9. *Let $m > n+1$ and $B^{max} > B_m > B_n$ with $\hat{B} > B_n$. There are two equilibria $p_n^*(B_n) = 1$ and $p_{m1}^*(B_m) \in (0,1)$, and $p_n^*(B_n) = 1$ and $p_{m2}^*(B_m) \in (0,1)$ with $p_{m1}^*(B_m) < p_{m2}^*(B_m)$ if $B_n < A_n|_{p_m=p_{m2}^*(B_m)}$, one equilibrium $p_n^*(B_n) = 1$ and $p_{m1}^*(B_m) \in (0,1)$ if $B_n \in [A_n|_{p_m=p_{m1}^*(B_m)}, A_n|_{p_m=p_{m2}^*(B_m)})$ and no equilibrium otherwise. Let $m > n+1$ and $B_m = B^{max}$. Then there exists one equilibrium $p_n^*(B_n) = 1$ and $p_m^*(B_m) \in (0,1)$ if $B_n < A_n|_{p_m=p_m^*(B_m)}$.*

Proof. The first part of the proof follows from Proposition 3 in Mavridis and Serena (2018). If $m > n+1$ then A_m reaches a maximum at $\hat{p}_m < 1$ and at $p_m = 1$ A_m becomes zero. Since $\hat{B} = A_m|_{p_m=p_m^{**}}$ and $p_m^{**} < \hat{p}_m$, $A_m = B_m$ has two roots, which we will call p_{m1}^* and p_{m2}^* and without loss of generality we will assume that $p_{m1}^* < p_{m2}^*$. It follows from Proposition 3 in Mavridis and Serena (2018) that if $B_n > A_n|_{p_m=p_{m1}^*}$ no equilibrium exists. If $B_n \in [A_n|_{p_m=p_{m1}^*}, A_n|_{p_m=p_{m2}^*})$ then only the equilibrium $p_n^* = 1$,$p_{m1}^* \in (0,1)$ exists. Otherwise, if $B_n < A_n|_{p_m=p_{m2}^*}$, there are two equilibria $p_n^* = 1$, $p_{m1}^* \in (0,1)$ and $p_n^* = 1$, $p_{m2}^* \in (0,1)$. For the second part notice that since B_m equals the unique maximum of A_m there is only one solution of $B^{max} = A_m$. □

Comparative Statics. We provide comparative statics on p_m^* of the "Partially Mixed" equilibrium of the form $p_n^* = 1$ and $p_m^* \in (0,1)$.

Proposition 10. *In the intervals for which A_m is increasing, p_m^* is increasing in B_m and decreasing otherwise. For $A_m \leq (\geq) A_n$, p_m^* is decreasing (increasing) in m. Finally if $m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1) \leq (\geq) 0$ then p_m^* is increasing (decreasing) in n.*

Proof. The first two statements follow from Proposition 3 in Mavridis and Serena (2018). The only part that needs some further analysis is the last statement. Notice that this quadratic inequality follows from Proposition 3 in Mavridis and Serena (2018) and that the roots of the quadratic equation $m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1) = 0$ are given by:

$$\tilde{p}_m = \frac{n(n-1)}{n(n-1) \pm \sqrt{n(n-1)(n(n-1) + m(m-2n-1))}}.$$

We discard the root with the negative sign in front of the square root because it becomes negative if $m-2n-1 > 0$ and greater than one otherwise. Let $m-2n-1 > 0$. If $p_m \leq \tilde{p}_m$ then $m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1) \leq 0$ holds and p_m^* is increasing in n. On the other hand if $m-2n-1 < 0$ the function $f(p_m) = m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1)$ is strictly concave and has a unique maximum. At the unique maximum the value of the function is: $n(n-1) + m(m-2n-1)$, which is greater than zero as long as $n(n-1) \geq m(2n+1-m)$. In this case if $p_m \leq \tilde{p}_m$ then $m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1) < 0$ holds and p_m^* is increasing in n. Otherwise if $n(n-1) < m(2n+1-m)$ then $m(m-2n-1)p_m^2 + 2n(n-1)p_m - n(n-1) < 0$ always holds and p_m^* is increasing in n. □

In this section we derived the full characterization of the "Pure" and "Partially Mixed" equilibria. Several differences emerge from the setting with symmetric benefits analyzed in Mavridis and Serena (2018). While in the symmetric setting of Mavridis and Serena (2018) the "Partially Mixed" is unique, in the present asymmetric setting there are two "Partially Mixed" (as clear from the propositions of this section, and also already from Figure 1). However, only one connects continuously with the unique "Pure". Furthermore, the same "Partially Mixed" also connects continuously to one "Totally Mixed", as we anticipated in Section 3 and we will further discuss in the next Section. Finally, while Mavridis and Serena (2018) analysed the "Totally Mixed" only numerically, we derive an analytical result in Proposition 11. The Proposition verifies Mavridis and Serena (2018)'s conjectures, arising from their numerical results.

4.3 "Totally Mixed" equilibria and cost-continuity refinement

Analytical findings. Propositions 1 to 10 fully characterized the "Pure" and "Partially Mixed" equilibria of the voting game. We are left to analyze the "Totally Mixed" equilibria. As discussed earlier in the paper, the mixing conditions for the m players is defined by:

$$A_m = B_m \tag{14}$$

and the mixing condition for the n players is defined by:

$$A_n = B_n \tag{15}$$

Since we have imposed the condition that all players within a group employ the same strategy, it suffices to focus on the mixing condition of an m player for every p_n, and that of an n player for every p_m, and analyze the intersections between these two in the (p_m, p_n)-space. These intersections are "Totally Mixed" equilibrium pairs (p_m^*, p_n^*), which are what we are after in this Subsection.

In contrast with the "Pure" and "Partially Mixed" cases, it is challenging to derive a general algebraic solution for equilibrium strategies of the "Totally Mixed" case, as expressions (14) and (15) are a system of two polynomial equations of arbitrary power. Instead, in order to analyze them we will use a number of indirect results about the space these equilibria lie on. For this it is useful to distinguish among the three cases: $B_m = B_n$, $B_m > B_n$, and $B_m < B_n$. Thus, by (14) and (15), these translate into $A_m = A_n$, $A_m > A_n$, and $A_m < A_n$. Analyzing $A_m = A_n$ will greatly help the analysis of the other two cases.

The set of points for which $A_m = A_n$ is depicted by the two black lines of Figure 3; the increasing and the decreasing one. These two black lines divide the (p_m, p_n)-space in four regions depending on the ranking of A_m and A_n. Keep in mind that the two mixing conditions are defined in the same space; dividing the space in these four regions will help us analyze how the intersections of the two mixing conditions behave.[16]

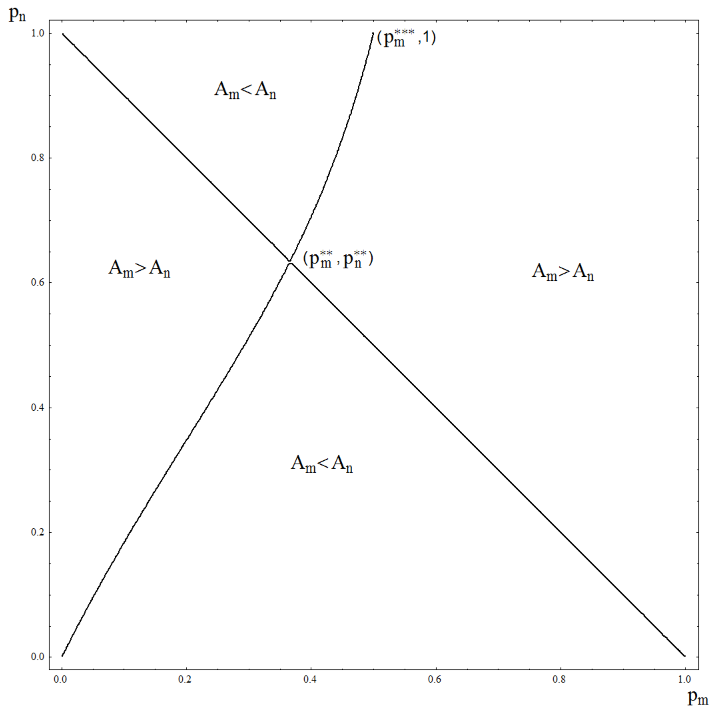

Figure 3: Set of points in the $(p_m), p_n)$-space according to whether $A_m \gtreqless A_n$ when $m = 3$ and $n = 2$.

In Appendix A we characterize the set $A_m = A_n$, through a series of lemmas. First, we find the four points where these two lines touch the edges of the (p_m, p_n)-space (Lemma 1). Second, we characterize the decreasing line connecting the top-left corner with the bottom-right corner (Lemma 2). Finally we characterize the increasing line (Lemmas 3

[16] At this stage it is interesting to compare our analysis with the one of Palfrey and Rosenthal (1983). In particular in Section 6 of Palfrey and Rosenthal (1983), they discuss "totally quasi-symmetric equilibria", which are what we call "Totally Mixed" equilibria. However they analyze two special cases, which in our notation are: i) $p_m = p_n$ and $m = n$ and ii) $p_m + p_n = 1$. In terms of our Figure 3 it means that they analyze equilibria that might arise along the two diagonals (in the case of the 45-degree line, they also assume $m = n$).

and 4).[17] We summarize this series of lemmas in Proposition 11.

Proposition 11. *The only points* $(p_m, p_n) \in (0,1)^2$ *satisfying condition*

$$A_m = A_n \tag{16}$$

are the points along the line $p_m + p_n = 1$ *and along a continuous line that goes from* $(0,0)$ *to* $(p_m^{***}, 1)$ *where* $p_m^{***} = \frac{n(n-1)}{n(n-1)+\sqrt{n(n-1)(m-n)(m-n+1)}}$.

Proof. See Appendix A. □

This result characterizes analytically what Mavridis and Serena (2018) found numerically, and described in their Figure 2. Furthermore, notice that in their Figure 2, the mixing conditions always cross in the set $A_m = A_n$, because they are interested only in the symmetric case of $B_m = B_n$. On the contrary, we are interested in the more general case where $B_m \gtreqless B_n$. For what concerns our *(Redistribution-Parametrized Example)*, as can be seen in (5) and (6), we have $B_m > B_n$, and thus all equilibria lie in one of the two regions where $A_m > A_n$ of Figure 3.

Numerical findings. Our numerical exercise build on the analytical results of sections 4.1-4.3. In fact, propositions 1-11 are necessary to look, when running numerical simulations, for "Totally Mixed" equilibria *only*, which satisfy the cost-continuity refinement, as our analytical results give us a full understanding of the behaviour of "Pure" and "Partially Mixed" equilibria. Our numerical exercise shows that the two mixing conditions (14) and (15) cross at most once in each of the four regions delimited by the set of points such that $A_m = A_n$ (see Figure 3). This implies that for every pair (B_m, B_n), we always have at most two "Totally Mixed" equilibria, one where $p_m^* + p_n^* < 1$ which we name "Totally Mixed 1", and one where $p_m^* + p_n^* > 1$ which we name "Totally Mixed

[17] Note that, as it will be explained in more detail later, the fact that the increasing line is in fact increasing is not needed. What is only needed is that it crosses the decreasing line once.

2".[18] Including the "Partially Mixed" equilibrium previously characterized, this shows that, all in all, we have at most three equilibria.

The crucial feature that emerges from our numerical exercise is that exactly one "Totally Mixed" equilibrium satisfies the cost-continuity refinement mentioned in Section 3. We now deliver the graphical intuition about uniqueness and cost-continuity.

In Figure 4 we depict in the (p_m, p_n)-space the conditions (14) and (15), where the red (blue) lines represent the mixing condition correspondence $A_n = B_n$ ($A_m = B_m$) for a $n-$ ($m-$) player for four values of B, $\{0.166, 0.333, 0.5, 0.61\}$.[19] In particular, we choose the third value to be the minimum B such that the "Totally Mixed 1" equilibrium disappears (bottom-left panel), and the fourth value to be the minimum B such that even the "Totally Mixed 2" equilibrium disappears. The black lines in Figure 4 are the set of (p_m, p_n) satisfying $A_m = A_n$, as in Figure 3.

The equilibrium on the right side of the panel is the "Totally Mixed 2" equilibrium, which exists as long as $B \leq 0.61$. This equilibrium cannot clearly be continuously connected to the "Partially Mixed" equilibrium, as we need p_i^* to go to 0 and at the same time p_j^* to be interior.[20] For this to happen we need that $p_m^* + p_n^* < 1$, which contradicts "Totally Mixed 2", which hence is ruled out by the cost-continuity refinement.

The equilibrium on the left side of the panel is the "Totally Mixed 1" equilibrium, which exists as long as $B \leq 0.5$. In particular, it converges to the "Partially Mixed" equilibrium $(p_m^*, p_n^*) = (0, 0.5)$ as $B \to 0.5$. Our numerical exercise shows that this continuous connection between the "Partially Mixed" and the "Totally Mixed 1" is a general property. We exploit the uniqueness of equilibrium under the cost-continuity refinement that we derive in our numerical exercise in order to discuss comparative statics and characterization of the *(Redistribution-Parametrized Example)*, to which we dedicate

[18] These definitions of "Totally Mixed 1" and "Totally Mixed 2" are in line with those of Section 3.

[19] Notice that Figure 4 is reminiscent to Figure 2 in Mavridis and Serena (2018). However their setting with symmetric payoffs yielded equilibria always lying along the increasing or decreasing line. In our setting with asymmetric payoffs the equilibria do not lie on these two lines.

[20] In terms of Figure 3 and Figure 4, as the value of B changes, the equilibria move inside the unitary box. One can see from the third panel of Figure 4 that for $B = 0.5$ the equilibria reach the edges of the box; namely, $(p_m, p_n) = (0, 0.5)$, and $(p_m, p_n) = (0.5, 1)$.

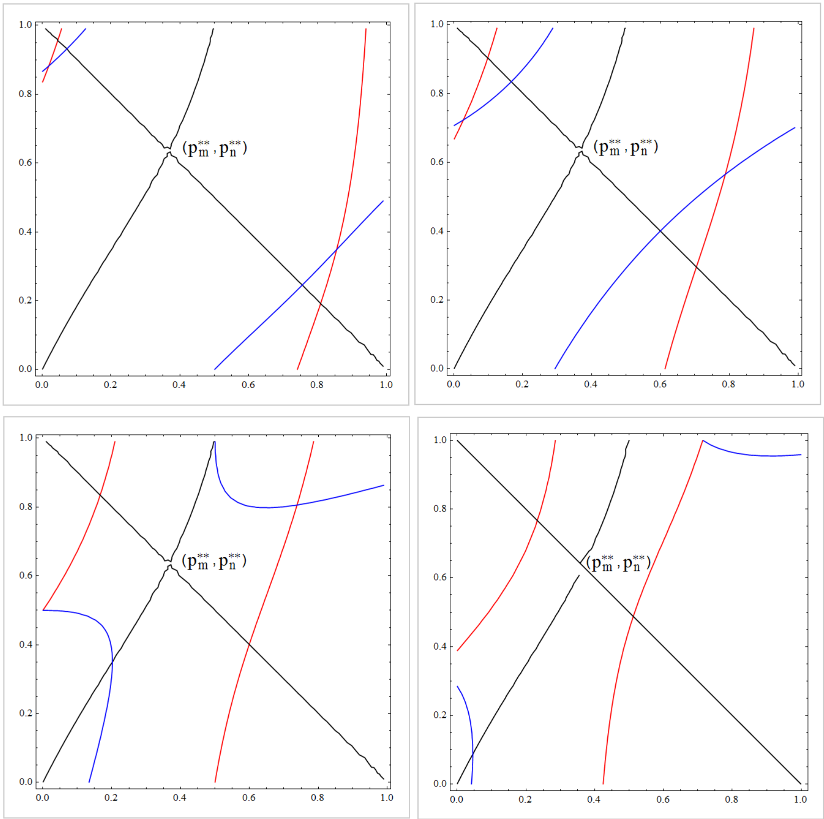

Figure 4: Mixing conditions correspondences in the (p_m, p_n)-space respectively for $B = \{0.166, 0.333, 0.5, 0.61\}$.

the next Section, so as to shed light on the redistribution trade-off spelled out in the Introduction.

5 Application - voting over redistribution of resources

From Proposition 4 we know that the "Partially Mixed" $p_m^*(B_m) = 0$ and $p_n^*(B_n) = 1 - B_n^{\frac{1}{n-1}}$ exist if and only if $B_m > nB_n - (n-1)B_n^{\frac{n}{n-1}}$. Recall from (5) and (6) that $B_m = 2c(n+m)/n$ and $B_n = 2c(n+m)/m$. Finally, recall that the notation for our *(Redistribution-Parametrized Example)* is $B_n = B$ and $B_m = \frac{m}{n}B$. Plugging these expressions into

$$B_m \geq nB_n - (n-1)B_n^{\frac{n}{n-1}}, \tag{17}$$

we obtain

$$\begin{aligned}\left(m-n^2\right)B+n(n-1)B^{\frac{n}{n-1}} &> 0\\ B &> \left(\frac{n^2-m}{n^2-n}\right)^{n-1}.\end{aligned}$$

And in fact, the lowest value of B for which the "Partially Mixed" exists is 0 when $n^2 \le m$, while when $n^2 > m$ it is $\left(\frac{n^2-m}{n^2-n}\right)^{n-1}$, which decreases in m. In fact, as we increase m, we see that such threshold moves to the left in the graph.

Notice that the "Partially Mixed" $p_m^*(B_m) = 0$ and $p_n^*(B_n) = 1 - B_n^{\frac{1}{n-1}}$ exists until $p_n^* \searrow 0$, which implies $1 - B_n^{\frac{1}{n-1}} = 0 \iff B_n = B = 1$. Hence, the highest B where the "Partially Mixed" exists is always equal to $B = 1$. Moreover, the mixing equilibrium probability $p_n^*(B_n) = 1 - B^{\frac{1}{n-1}}$ in the "Partially Mixed" has the same graph in the four plots, as the functional form is the same and n does not change. Only its existence region becomes bigger as m increases.

Therefore, if the "Partially Mixed" still exists as B goes to 0 (equivalently, c goes to 0), then the "Partially Mixed" and "Pure" together form the unique equilibrium without any need for cost-continuity, and the poor players will never vote for any $c > 0$. Notice that the second term of the right-hand side of (17) goes to 0 faster than the other terms in the inequality as B's go to zero, and thus in the limit it is negligible. Then, sufficiently close to 0, we are left with only $B_m > nB_n$. By plugging the expressions for B_m and B_n we get:

$$m \ge n^2. \tag{18}$$

This is a necessary and sufficient condition for $p_m^* = 0$ to hold in the unique equilibrium for any $c > 0$. If the number of poor voters is sufficiently low ($m < n^2$), the poor might vote and redistribution has a chance of winning. However, if their number is sufficiently large ($m \ge n^2$), the poor are doomed to lose the election.

Comparative Statics. How does a change of m affect (p_m^*, p_n^*)? We answer with the support of Figure 5. We fix $n = 3$, and set m so as to initially have an economy with similar group size ($m = 4$, top-left), and gradually increase the group size of m ($m = 5$, top-right, and $m = 6$, bottom-left), until we hit threshold $m = n^2$ ($m = 9$, bottom-right). When we hit this threshold condition (18) is satisfied and the "Totally Mixed 1" equilibrium (which survives cost-continuity) disappears, and we are left with the "Partially Mixed" and "Pure" only.[21]

Consider $n = 3$ and $m = 4$. Since the economy has (slightly) more poor than rich players, the average resource level is closer to the resources of a poor player than to the resources of a rich player, thus if full redistribution wins, the individual loss of a single rich player is greater than the individual gain of a single poor player. For this reason, an n rich player is willing to vote for greater B's than an m poor player. In other words, a rich has more at stake than a poor, and thus is willing to face a greater cost of voting. Therefore, p_n^* turns positive for greater values of B than p_m^* does, as we can see in Figure 5.

Consider an increase of m ($n = 3$ and $m = 5$, or 6). This has the effect of sharpening the asymmetry in willingness to face the cost of voting between rich and poor: in fact, now, p_n^* turns positive for even greater B's (the rich has even more at stake to lose in case of full redistribution), while p_m^* turns positive for even lower B's (the poor has even less at stake to win in case of full redistribution). This widens the "Partially Mixed" region (see Figure 5).

If the increase in m reaches the threshold when $m = n^2$ ($n = 3$ and $m = 9$), the poor has so little at stake that she is nowhere willing to face the cost of voting with positive probability in equilibrium.[22] A further increase in m would still imply $p_m^* = 0$ everywhere, and further increases the willingness to vote of the rich (i.e., p_n^* increases for any given B, and p_n^* turns positive for greater B's).

[21]This possibility that members of the majority never vote in the unique continuous equilibrium did not emerge from the discussion of the special case of $m = 3$ and $n = 2$ in Figure 1. In fact, when $m = 3$ and $n = 2$, $m < n^2$.

[22]Remember that if $m \geq n^2$ the equilibrium is unique without the need for continuity selection.

The direct interpretation of this result is that the greater is the share of poor in an economy, the less likely is redistribution of resources to be the outcome of a democratic process (and if $m \geq n^2$ this probability is zero).

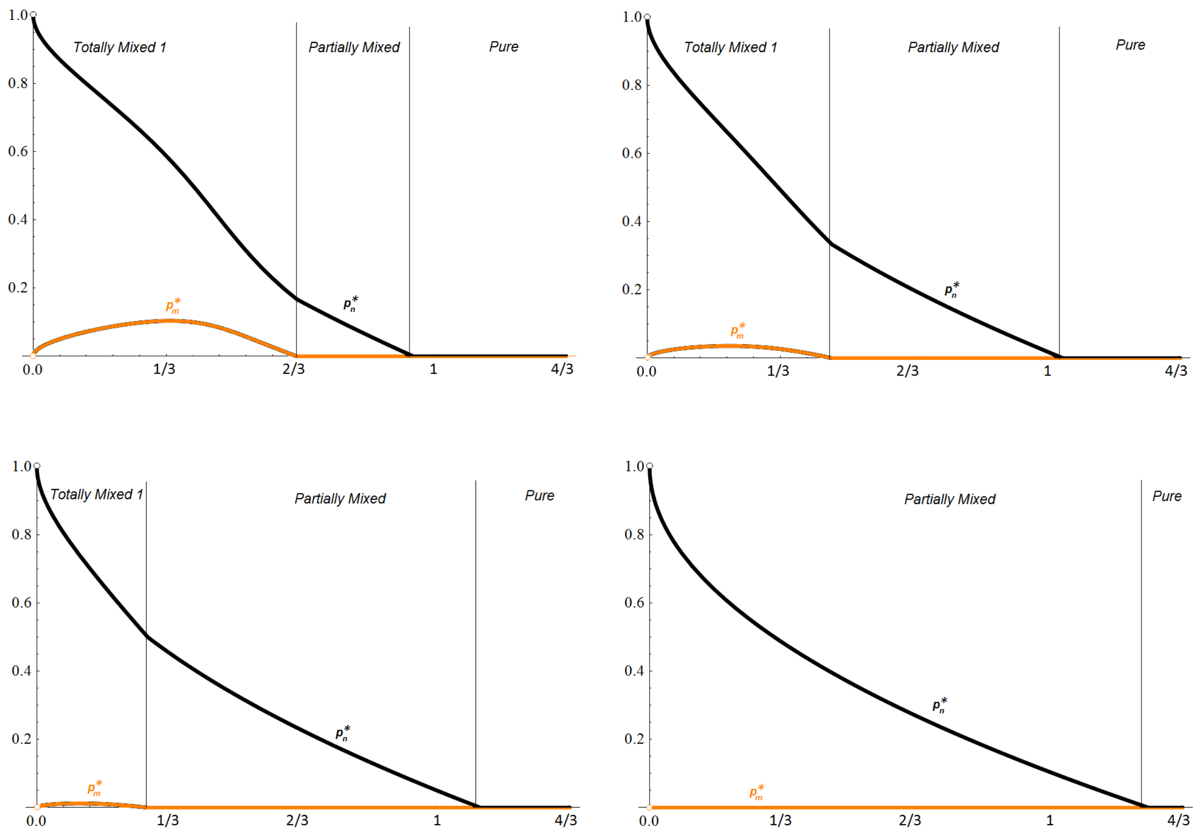

Figure 5: Effects of increasing m, keeping $n = 3$, on voting probabilities in the unique continuous equilibrium. First row left panel $m = 4$, right panel $m = 5$. Second row, left panel $m = 6$ right panel $m = 9$.

Asymmetric costs and different benefits. The key parametrization of the redistribution example we kept throughout the paper is $B_m/B_n = m/n$. First, the key contribution of the present paper as opposed to the existing literature (i.e., Nöldeke and Pena, 2016; Mavridis and Serena, 2018) is that we allow $B_m/B_n \neq 1$. The choice of keeping as a redistribution-parametrized running example the parametrization $B_m/B_n = m/n$ is for illustration purposes; the redistribution example, implying the simple relation $B_m/B_n = m/n$, allows to draw tight conclusions of Section 5.

Second, the parametrization $B_m/B_n = m/n$ could be achieved not only assuming $c_m = c_n = c$ and the B_m and B_n specified in (5) and (6), as we did, but also allowing for asymmetry in the costs $c_m = k_1 \times c_n$ and rescaling the benefits such that $B_m/B_n = k_1 \times (m/n)$. Hence, our results under the parametrization of Section 5 hold also for this case of asymmetric costs. In such case, the results of Section 5 carry over to asymmetric costs if the two policies over which players are called to vote are two levels of redistribution resulting in a ratio of benefits $B_m/B_n = k_1 \times (m/n)$, rather than full- and no-redistribution.

Third, and most importantly, one may want to model redistribution options and asymmetries of costs such that the ratio B_m/B_n takes a different value than m/n; say $B_m/B_n = k_2$. Our results (i.e., propositions 1-11) are general and encompass any such k_2, and not only $k_2 = m/n$ as analyzed in Section 5. Fixing $B_n = B$ again for visual purposes such that $B_m = k_2 \times B$ (where k_2 no longer takes value m/n), a k_2 smaller than m/n but still greater than 1 would produce the same qualitative effects of a change in m depicted in Figure 5. If k_2 takes value smaller than 1, the results are "as if" we swap the identity of type m and type n. Think of $k_2 = n/m$. Then Figure 5 would hold the way it is just swapping identities.

6 Conclusions

We study a standard pivotal-voter model under majority rule, with two rival groups of players, each preferring one of two public policies and simultaneously deciding whether to cast a costly vote. We further the existing literature, in particular Palfrey and Rosenthal (1983), Nöldeke and Peña (2016) and Mavridis and Serena (2018), by allowing the benefit of the favorite public policy to differ across groups. We impose an intuitive refinement, namely that voting probabilities are continuous in the cost of voting to pin down a unique equilibrium. The unique cost-continuous equilibrium depends on a key threshold that compares the sizes of the two groups.

Besides cost-continuity, it would be interesting to study alternative refinements, such as refinements based on the payoffs and preferences of players.[23] Note, however, that the exercise of checking whether an equilibrium which is the best for members of both groups exists and finding it, is not a tractable one as a general closed-form solution is lacking for equilibrium probabilities of voting. Simplifying the setup might yield the desired tractability that allows a thorough analysis of individuals' payoffs. For instance, the setup of Nöldeke and Peña (2016), which assume that both the number of members and the benefit from the favored alternative winning the election is identical across groups, allows for an elegant characterization of a number of analytical properties of individual's probability of being pivotal. However, here, we focused on the case where both number of members and benefits could vary across groups.

[23]We thanks an anonymous referee for pointing this out.

Appendix

We prove Proposition 11 by way of the following lemmata. See Figure 3.

Lemma 1. *The only points satisfying (16) and* $(p_m, p_n) \in \{0,1\}^2$ *are:* $(0,0)$, $(0,1)$, $(1,0)$, *and* $(p_m^{***}, 1)$, *with* $p_m^{***} = \frac{n(n-1)}{n(n-1)+\sqrt{n(n-1)(m-n)(m-n+1)}}$. *Also,* $p_m^{***} = 1$ *iff* $m = n$.

Proof. By continuity of A_m and A_n in p_m and p_n, in order to analyze the behavior of A_m and A_n in $(p_m, p_n) \in \{0,1\}^2$ we can compute the following limits for A_m

$$
\begin{aligned}
\lim_{p_m \to 0} A_m &= np_n(1-p_n)^{n-1} + (1-p_n)^n \\
\lim_{p_m \to 1} A_m &= \begin{cases} p_n^n + np_n^{n-1}(1-p_n) & \text{if } m = n \\ \binom{m-1}{n} p_n^n & \text{if } m = n+1 \\ 0 & \text{if } m > n+1 \end{cases} \\
\lim_{p_n \to 0} A_m &= (1-p_m)^{m-1} \\
\lim_{p_n \to 1} A_m &= \binom{m-1}{n-1} p_m^{n-1}(1-p_m)^{m-n} + \binom{m-1}{n} p_m^n (1-p_m)^{m-n-1}
\end{aligned}
$$

and for A_n

$$
\begin{aligned}
\lim_{p_m \to 0} A_n &= (1-p_n)^{n-1} \\
\lim_{p_m \to 1} A_n &= \begin{cases} p_n^{n-1} & \text{if } m = n \\ 0 & \text{if } m > n \end{cases} \\
\lim_{p_n \to 0} A_n &= m(1-p_m)^{m-1} + (1-p_m)^m \\
\lim_{p_n \to 1} A_n &= \binom{m}{n} p_m^n (1-p_m)^{m-n} + \binom{m}{n-1} p_m^{n-1}(1-p_m)^{m-n+1}
\end{aligned}
$$

From the above,

- if $p_m = 0$, (16) holds iff $p_n = 0$ or $p_n = 1$
- if $p_m = 1$, (16) holds iff $p_n = 0$ or $p_n = 1$ and $m = n$
- if $p_n = 0$, (16) holds iff $p_m = 0$ or $p_m = 1$

- if $p_n = 1$, (16) is equivalent to

$$\binom{m-1}{n-1} p_m^{n-1}(1-p_m)^{m-n} + \binom{m-1}{n} p_m^{n}(1-p_m)^{m-n-1} = \binom{m}{n} p_m^{n}(1-p_m)^{m-n} + \binom{m}{n-1} p_m^{n-1}(1-p_m)^{m-n+1}$$

$$\binom{m-1}{n-1}(1-p_m) + \binom{m-1}{n} p_m = \binom{m}{n} p_m(1-p_m) + \binom{m}{n-1}(1-p_m)^2 \tag{19}$$

If $m = n$, (19) boils down to

$$1 - p_m = p_m(1-p_m) + m\,(1-p_m)^2$$

whose unique solution is $p_m = 1$.

If $m > n$, (19) boils down to

$$\frac{(1-p_m)}{m-n} + \frac{p_m}{n} = \frac{mp_m(1-p_m)}{n(m-n)} + \frac{m(1-p_m)^2}{(m-n)(m-n+1)}$$

Solving the simple polynomial in the last expression we see that p_m^{***} is indeed one of its two roots (the second root has to be discarded since it is greater than 1). □

Lemma 2. *Equation $p_m + p_n = 1$ solves $A_m = A_n$ $\forall (p_m, p_n) \in [0,1]^2$.*

Proof. From (9) plug A_m and A_n into (16), simplify for $(1-p_m)^m(1-p_n)^n$, and use $p_n = 1 - p_m$ to obtain

$$\sum_{s=0}^{n} \binom{m-1}{s}\binom{n}{s} p_m^{s-s}(1-p_m)^{-s-1+s} + \sum_{s=0}^{n-1} \binom{m-1}{s}\binom{n}{s+1} p_m^{s-s-1}(1-p_m)^{-s-1+s+1}$$
$$= \sum_{s=0}^{n-1} \binom{m}{s}\binom{n-1}{s} p_m^{s-s-1}(1-p_m)^{-s+s} + \sum_{s=0}^{n-1} \binom{m}{s+1}\binom{n-1}{s} p_m^{s+1-s-1}(1-p_m)^{-s-1+s}$$

$$p_m \sum_{s=0}^{n} \binom{m-1}{s}\binom{n}{s} + (1-p_m)\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}$$
$$= (1-p_m)\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s} + p_m \sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}$$

$$p_m \sum_{s=0}^{n} \binom{m-1}{s}\binom{n}{n-s} + (1-p_m)\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{n-s-1}$$
$$= (1-p_m)\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{n-s-1} + p_m \sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{n-s-1}$$

$$p_m\binom{m+n-1}{n} + (1-p_m)\binom{m+n-1}{n-1} = (1-p_m)\binom{m+n-1}{n-1} + p_m\binom{m+n-1}{n}$$
$$0 = 0$$

where in the second-to-last step we used the symmetry rule for binomial coefficients, and in the last step we used Vandermonde's identity.[24] □

Next we characterize the set of points $A_m = A_n$ that are depicted by an increasing line in the (p_m, p_n)-space by means of two lemmas. In Lemma 3 we show that there exists a point (p_m^{**}, p_n^{**}) along the decreasing line which divides the neighborhoods of the decreasing line into two parts:

1. The first part is the one connecting (p_m^{**}, p_n^{**}) and $(1, 0)$, where we prove that increasing p_m (i.e., moving to the right of the line), increases A_m *more* than A_n. Since exactly along the line $A_m = A_n$, this result implies that to the right of the segment connecting (p_m^{**}, p_n^{**}) and $(1, 0)$ we have $A_m > A_n$ and to its left we have $A_m < A_n$.

2. The second part is the one connecting $(0, 1)$ and (p_m^{**}, p_n^{**}), where we prove that increasing p_m (i.e., moving to the right of the line), increases A_m *less* than A_n. Since exactly along the line, $A_m = A_n$, this result implies that to the right of

[24]Vandermonde's identity states that $\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$ for $m, n, r \in \mathbb{N}_0$.

the segment connecting $(0,1)$ and (p_m^{**}, p_n^{**}) we have that $A_m < A_n$ and to its left $A_m > A_n$.

Lemma 3. *There exists a unique pair* $(p_m^{**}, p_n^{**}) \in (0,1)^2$ *with* $p_m^{**} + p_n^{**} = 1$ *such that*

$$\left.\frac{\partial A_m}{\partial p_m}\right|_{p_m+p_n=1} > \left.\frac{\partial A_n}{\partial p_m}\right|_{p_m+p_n=1} \text{ iff } p_m > p_m^{**} \text{ (or equivalently } p_n < p_n^{**}\text{)}$$

Also, if $m = n$*, then* $p_m^{**} = p_n^{**} = \frac{1}{2}$*, and if* $m > n$*, then* $p_m^{**} \in (0, \frac{1}{2})$ *and* $p_n^{**} \in (\frac{1}{2}, 1)$*. In particular,*

$$p_m^{**} = \frac{n(n-1)}{n(n-1) + \sqrt{m(m-1)n(n-1)}} \text{ and } p_n^{**} = 1 - p_m^{**}$$

Proof. For notation simplicity and for the sake of space we define the following

$$\begin{aligned} \tilde{p}_{s,m} &= \left(\frac{p_m}{1-p_m}\right)^s \\ \tilde{p}_{s,n} &= \left(\frac{p_n}{1-p_n}\right)^s \end{aligned}$$

Then,

$$\left.\frac{\partial A_m}{\partial p_m}\right|_{p_m+p_n=1} > \left.\frac{\partial A_n}{\partial p_m}\right|_{p_m+p_n=1}$$

$$\sum_{s=0}^{n} \binom{m-1}{s}\binom{n}{s}\tilde{p}_{s,m}\tilde{p}_{s,n}\frac{s+p_m}{p_m(1-p_m)^2} + \left.\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}\tilde{p}_{s,m}\tilde{p}_{s,n}\frac{s+p_m}{p_m(1-p_m)^2}\frac{p_n}{1-p_n}\right|_{p_m+p_n=1} >$$

$$\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}\tilde{p}_{s,m}\tilde{p}_{s,n}\frac{s}{p_m(1-p_m)(1-p_n)} + \left.\sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}\tilde{p}_{s,m}\tilde{p}_{s,n}\frac{s+1}{(1-p_m)^2(1-p_n)}\right|_{p_m+p_n=1}$$

by noticing that $p_m + p_n = 1$ implies $\tilde{p}_{s,m}\tilde{p}_{s,n} = 1$ the above inequality simplifies to

$$\sum_{s=0}^{n} \binom{m-1}{s}\binom{n}{s}\frac{s+p_m}{p_m(1-p_m)^2} + \sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}\frac{s+p_m}{p_m^2(1-p_m)} >$$
$$\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}\frac{s}{p_m^2(1-p_m)} + \sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}\frac{s+1}{p_m(1-p_m)^2}$$

$$\sum_{s=0}^{n} \binom{m-1}{s}\binom{n}{s}p_m(s+p_m) + \sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}(1-p_m)(s+p_m) >$$
$$\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}(1-p_m)s + \sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}p_m(s+1)$$

Note that some summands in the above inequality contain s only in the binomial coefficients. By applying to these terms the same procedure at the end of Lemma 2 (i.e. symmetry rule for binomial coefficients and Vandermonde's identity), we get

$$p_m\sum_{s=0}^{n}\binom{m-1}{s}\binom{n}{s}s + p_m^2\binom{m+n-1}{n}$$
$$+ (1-p_m)\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}s + p_m(1-p_m)\binom{m+n-1}{n-1} >$$
$$(1-p_m)\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}s + p_m\sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}s + p_m\binom{m+n-1}{n}$$

$$p_m\sum_{s=0}^{n}\binom{m-1}{s}\binom{n}{s}s + (1-p_m)\sum_{s=0}^{n-1}\binom{m-1}{s}\binom{n}{s+1}s >$$
$$(1-p_m)\sum_{s=0}^{n-1}\binom{m}{s}\binom{n-1}{s}s + p_m\sum_{s=0}^{n-1}\binom{m}{s+1}\binom{n-1}{s}s + p_m(1-p_m)\left[\binom{m+n-1}{n} - \binom{m+n-1}{n-1}\right]$$

we now analyze the summations left (containing s not only in the binomial coefficient), and use the fact that $\sum_{s=0}^{n}\binom{m}{s}\binom{n}{s}s = n\binom{m+n-1}{n}$ and that $\sum_{s=0}^{n}\binom{m}{s}\binom{n}{s+1}s = m\binom{m+n-1}{n-2}$,[25] and get

$$np_m\binom{m+n-2}{n} + (m-1)(1-p_m)\binom{m+n-2}{n-2} >$$
$$(n-1)(1-p_m)\binom{m+n-2}{n-1} + (n-1)p_m\binom{m+n-2}{n} + p_m(1-p_m)\left[\binom{m-n-1}{n} - \binom{m-n-1}{n-1}\right]$$

$$(m-1)(1-p_m)\binom{m+n-2}{n-2} >$$
$$(n-1)(1-p_m)\binom{m+n-2}{n-1} - p_m\binom{m+n-2}{n} + p_m(1-p_m)\left[\binom{m-n-1}{n} - \binom{m-n-1}{n-1}\right]$$

and simplifying by $\frac{(m+n-2)!}{(m-2)!(n-2)!}$ we get

$$\frac{1-p_m}{m} > \frac{1-p_m}{m-1} - \frac{p_m}{n(n-1)} + p_m(1-p_m)\frac{(m-n)(m+n-1)}{m(m-1)n(n-1)}$$
$$-n(n-1)(1-p_m) > -m(m-1)p_m + p_m(1-p_m)(m-n)(m+n-1)$$
$$(m-n)(m+n-1)p_m^2 + 2n(n-1)p_m - n(n-1) > 0$$
$$p_m > \frac{n(n-1)}{n(n-1)+\sqrt{m(m-1)n(n-1)}} = p_m^{**}$$

If $m = n$ it is trivial to see that $p_m^{**} = \frac{1}{2}$. But notice also that p_m^{**} decreases in m, and hence by $m > n$, $p_m^{**} \in (0, \frac{1}{2})$ and $p_n^{**} \in (\frac{1}{2}, 1)$. □

[25]

$$\begin{aligned}
\sum_{s=0}^{n}\binom{m}{s}\binom{n}{s}s &= \sum_{s=0}^{n}\binom{m}{s}\frac{n!}{s!(n-s)!}s \\
&= \sum_{s=1}^{n}\binom{m}{s}\frac{n!}{(s-1)!(n-s)!} = \sum_{s=1}^{n}\binom{m}{s}\frac{n!}{(s-1)!(n-1-s+1)!} \\
&= \sum_{s=1}^{n}\binom{m}{s}\frac{(n-1)!}{(s-1)!((n-1)-(s-1))!}n = \sum_{s=0}^{n}\binom{m}{s}\binom{n-1}{s-1}n \\
&= \sum_{s=0}^{n}\binom{m}{s}\binom{n-1}{n-s}n = n\binom{m+n-1}{n}
\end{aligned}$$

Where the last equality follows from Vandermonde's identity. The calculations for the other summation are similar.

Lemma 4 concludes the characterization of the increasing line.

Lemma 4. *There exists a unique and continuous line in the* (p_m, p_n)*-space which satisfies* $A_m = A_n$ *and connects* $(0, 0)$ *and* $(p_m^{***}, 1)$ *Furthermore, this line crosses the* $p_m + p_n = 1$ *line once at* (p_m^{**}, p_n^{**}).

Proof. Lemma 2 establishes that the decreasing line connects two out of the four points satisfying $A_m = A_n$ along the edges. The line connecting the remaining two points is continuous and by Lemma 3 crosses the decreasing line once, at (p_m^{**}, p_n^{**}). □

References

[1] Acemoglu, D, and Robinson, J. 2000. "Why Did the West Extend the Franchise? Democracy, Inequality and Growth in Historical Perspective," *Quarterly Journal of Economics*, 115(4), 1167-1199.

[2] Acemoglu, D, and Robinson, J. 2012. "Why Nations Fail," *Crown Publishing Group.*

[3] Borgers, T. 2004. "Costly Voting," *American Economic Review*, 94(1), 57-66.

[4] DellaVigna, S., List, J. A., Malmendier, U., Rao, G., 2017. "Voting to Tell Others," *Review of Economic Studies,* 84 (1), 143–181.

[5] Downs, A. 1957. "An Economic Theory of Democracy," *New York: Harper and Row.*

[6] Gerber, A. S., Green, D. P., Larimer, C., 2008. "Social Pressure and Voter Turnout: Evidence from a Large-Scale Field Experiment." *American Political Science Review* 102 (1), 16

[7] Gerber, A. S., Huber, G. A., Doherty, D., Dowling, C. M., 2016. "Why People Vote: Estimating the Social Returns to Voting." *British Journal of Political Science* 46 (2), 241–264.

[8] Großer, J., Schram, A. J. H. C., 2010. "Public Opinion Polls, Voter Turnout: An Experimental Study." *American Journal of Political Science* 54 (3), 700–717

[9] Kalandrakis, T., 2007. "On participation games with complete information," *International Journal of Game Theory,* 35 (3), 337–352.

[10] Kalandrakis, T., 2009. "Robust rational turnout"' *Economic Theory,* 41 (2), 317–343.

[11] Lyytikäinen, T. and Tukiainen, J. 2019. "Are Voters Rational?," *European Journal of Political Economy*, 59, 230-242.

[12] Mas-Colell, A., Whinston, M. D., and Green, J. R. 1995. Microeconomic theory (Vol. 1). New York: Oxford university press.

[13] Mavridis, C., and Serena, M. 2018. "Complete information pivotal-voter model with asymmetric group size," *Public Choice*, 177(1-2), 53-66.

[14] Meltzer, A. H. and Richard, S. F. 1981. "A Rational Theory of the Size of Government," *Journal of Political Economy*, 89(5), 917-927.

[15] Morton, R. B., Muller, D., Page, L., Torgler, B., 2015. "Exit polls, turnout, and bandwagon voting: Evidence from a natural experiment," *European Economic Review* 77, 65–81.

[16] Nöldeke, G., and J Peña, 2016. "The Symmetric Equilibria of Symmetric Voter Participation Games with Complete Information," *Games and Economic Behavior*, 99, 71-81.

[17] Palfrey, T. R., and Pogorelskiy, K., 2019. "Communication among voters benefits the majority party," *The Economic Journal*, 129(618), 961-990.

[18] Palfrey, T. R., and Rosenthal, H. 1983. "A strategic calculus of voting," *Public Choice*, 41(1), 7-53.

[19] Palfrey, T. R., and Rosenthal, H. 1985. "Voter Participation and Strategic Uncertainty," *The American Political Science Review*, 79(1), 62-78.

[20] Pogorelskiy, K., 2020. "Correlated Equilibria in Voter Turnout Games." *Forthcoming, Games and Economic Behavior.*

[21] Taylor, C. R., and Yildirim, H. 2010. "A unified analysis of rational voting with private values and group-specific costs," *Games and Economic Behavior*, 70(2), 457-471.

[22] Wiese, R., and Jong-A-Pin, R. 2017. "Expressive voting and political ideology in a laboratory democracy," *European Journal of Political Economy*, 50, 54-74.